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6

Topics in Quantitative Analysis of Social Protection Systems

Kenichi Hirose

Social Security Department

INTERNATIONAL LABOUR OFFICE GENEVA

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Hirose, Kenichi (hirose@ilo.org)

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Foreword

This discussion paper series was conceived as a market place of ideas where social protection professionals could air their views on specific issues in their field. Topics may range from highly technical aspects of quantitative analysis to aspects of social protection planning, governance and politics. Authors may come from within the ILO or be independent experts, as long as they have "something to tell" concerning social protection and are not afraid to speak their mind. All of them contribute to this series in a personal capacity - not as representatives of the organisations they belong to. The views expressed here are thus entirely personal, they do not necessarily reflect the views of the ILO or other organisations. The only quality requirements are that the papers either fill a gap in our understanding of the functioning of national social protection or add an interesting aspect to the policy debates.

The ILO believes that a worldwide search for a better design and management of social protection is a permanent process that can only be advanced by a frank exchange of ideas. This series is thought to be a contribution to that process and to the publicizing of new ideas or new objectives. It thus contributes to the promotion of social security which is one of the ILO's core mandates.

Preface

> Nevertheless, on matters of detail the statistical and mathematical methods evolved by modern economics can be of very great service in the planning of development, provided that they are thoroughly well scrubbed to get the metaphysical concepts cleaned off them.
>
> (Joan Robinson, Economic Philosophy)

> Discussions of economic policy, if they are to be productive in any practical sense, necessarily involve quantitative assessments of the way proposed policies are likely to affect resource allocation and individual welfare.
>
> (Robert E. Lucas Jr., Models of Business Cycles)

This volume comprises ten papers dealing with quantitative methods in social protection. These papers have emerged from occasional technical notes on various issues which I encountered in the course of my work in the Social Security Department of the ILO since 1994. The papers are roughly classified into four subjects, but each paper is written self-consistently. Those readers who are interested in mathematical foundations may find it helpful to refer to the last paper.

As these papers have been produced over a long period of time, the style and level of mathematics employed are different. Some papers appear to be completely different from their original notes due to subsequent generalization and sophistication. However, I would like to stress that all papers are motivated by problems concerning the application of quantitative analysis of social security systems in developing countries.

It is intended that this work will be integrated into a comprehensive treatise which forms a volume in the ILO/ISSA publication series Quantitative methods in social protection. This discussion paper was prepared to serve as a precursory draft asking for perusal by a wide range of people. Any comments and suggestions on these papers are welcome and should be addressed to the author (email: hirose@ilo.org).

These papers would not have existed in the style presented here without suggestions and encouragement from many colleagues and friends, whose contributions are acknowledged in the bibliographical notes. I would like to thank all of them without implicating them in errors which remain. Finally, allow me to dedicate this work to the memory of Jan Stoekenbroek.

Geneva, September 1999

Kenichi Hirose

Note added in the fourth LaTeX printing (Manila, December 2004):

From this printing, this paper will be distributed electronically. I have taken this opportunity to make some minor modifications and corrections. I am grateful to those who have contributed their comments. Let me extend my gratitude to the memory of Masanori Akatsuka.

Contents

Chapter 1

Financial Indicators and Systems of Financing of Social Security Pensions

A generalization of the concept of scaled premium

One of the objectives of the actuarial valuation of social security pension schemes is to set out the future contribution rates which ensure the long-term solvency of the schemes. The requirements for the long-term solvency are, in many cases, formulated in terms of financial indicators. In this paper, we examine the implications of three financial indicators and develop premium formulae which satisfy the required financial conditions.

1. Financial Indicators

1.1 Preliminaries

This paper focuses on a defined benefit pension scheme which is financed from two sources, namely contributions from workers' payroll and return on investment of the reserves.

Let

F_t	:	Reserve at the end of the year t.
I_t	:	The total income during year t (including investment income).
C_t	:	Contributions collected during year t (excluding investment income).
R_t	:	The return on investment of the reserve during year t.
E_t	:	The total expenditure incurred during year t, consisting of the total benefit payment B_t and the administrative expenses A_t (i.e., $E_t = B_t + A_t$).
S_t	:	The total contributory earnings during year t.
P_t	:	Contribution rate in year t.
i	:	The rate of return on investment of the reserve (assumed constant over time).

For simplicity, expenditure and contributions are assumed to occur at the middle of each year.

Between the above-defined accounting items, the following equations hold:

$$I_t = C_t + R_t, \tag{1}$$

$$R_t = (\sqrt{1+i} - 1)(C_t - E_t) + iF_{t-1}, \tag{2}$$

$$\Delta F_t = F_t - F_{t-1} = I_t - E_t, \tag{3}$$

$$C_t = p_t S_t. \tag{4}$$

From equations(1)-(4), it follows that

$$F_t = (1+i)F_{t-1} + \sqrt{1+i} \cdot (C_t - E_t). \tag{5}$$

Equation(5), seen as the recursion formula of the sequence $\{F_t\}$, describes the evolution of the reserve from one year to the next.

If the interest rate and future figures of total contributory earnings, expenditure and contribution rates are given, then the contribution income, the interest income and the total income are determined by equations (1) to (4). Furthermore, starting with the reserve in the initial year, the reserve in the future years can be obtained by applying formula (5) successively. Consequently, the independent variables for the calculation of the future reserve are S_t , E_t , p_t , F_0 and i.

1.2 Actuarial implications of financial indicators

Standardized financial indicators are useful for analyzing the long-term transition of the financial status and comparing financial positions of different schemes, rather than the expenditure and total contributory earnings expressed in nominal terms. In this paper, we device three financial indicators, called the pay-as-you-go cost rate, the reserve ratio and the balance ratio, and study their basic properties.

(i) **Pay-as-you-go (PAYG) cost rate**

The PAYG cost rate in year t, denoted by $PAYG_t$, is defined as:

$$PAYG_t = \frac{E_t}{S_t}. \tag{6}$$

This indicator is considered as the contribution rate needed for the payment of current pensions on the assumption that the costs are financed from current total contributory earnings.

Since E_t is the sum of the total benefit payment B_t and the administrative expenses A_t, the PAYG cost rate is written as the sum of the net PAYG rate (or PAYG benefit rate), denoted by c_t and the PAYG administrative cost rate, denoted by m_t.

$$PAYG_t = \frac{E_t}{S_t} = \frac{B_t}{S_t} + \frac{A_t}{S_t} = c_t + m_t. \tag{7}$$

Further, the net PAYG rate is written as a product of two factors:

$$c_t = d_t \cdot r_t, \tag{8}$$

where d_t is called the "demographic dependency ratio" and r_t is called the "replacement ratio", defined by

d_t = (number of pensioners in year t) / (number of active contributors in year t),

r_t = (average pension in year t) / (average contributory earnings in year t).

Equation (8) implies that if the relative number of active contributors to pensioners is lower or the relative pension level as compared with the average contributory earnings is higher, then the net PAYG rate is higher. If the replacement ratio does not change significantly over time, the demographic ratio is considered to be the determinant of the long-term development of the net PAYG rate. A rapid progress of the population ageing, therefore, may result in a rapid increase in the net PAYG rate.

(ii) **Reserve ratio**

The reserve ratio in year t, denoted by at, is defined as[1]

$$a_t = \frac{F_{t-1}}{E_t}. \tag{9}$$

This represents the relative level of the reserve at the beginning of a year measured by the expenditure incurred during that year. The reserve ratio may be considered as an indicator of the short-term solvency of the scheme. For example, if pensions are paid monthly, the scheme should retain a reserve at least equal to one month's benefit at the beginning of each month, which is approximately 9% of the annual expenditure. Therefore, the level of the reserve ratio required for this purpose is 0.09.

(iii) **Balance ratio**

The balance ratio in year t, denoted by b_t , is defined as[2]

$$b_t = \frac{E_t - C_t}{R_t}. \tag{10}$$

This financial indicator represents the amount of expenditure in excess of the current contribution income expressed as a percentage of the expected interest income. For a given year, if the contribution income is more than the expenditure, the interest income on the fund does not need to be used; conversely, if the contribution income is less than the expenditure, the interest income must be used to meet the expenditure.

[1] Some actuaries adopt a different definition: $a_t = F_t/E_t$. This definition reflects the view that the reserve level should be measured at the end of the year instead of at the beginning of the year.

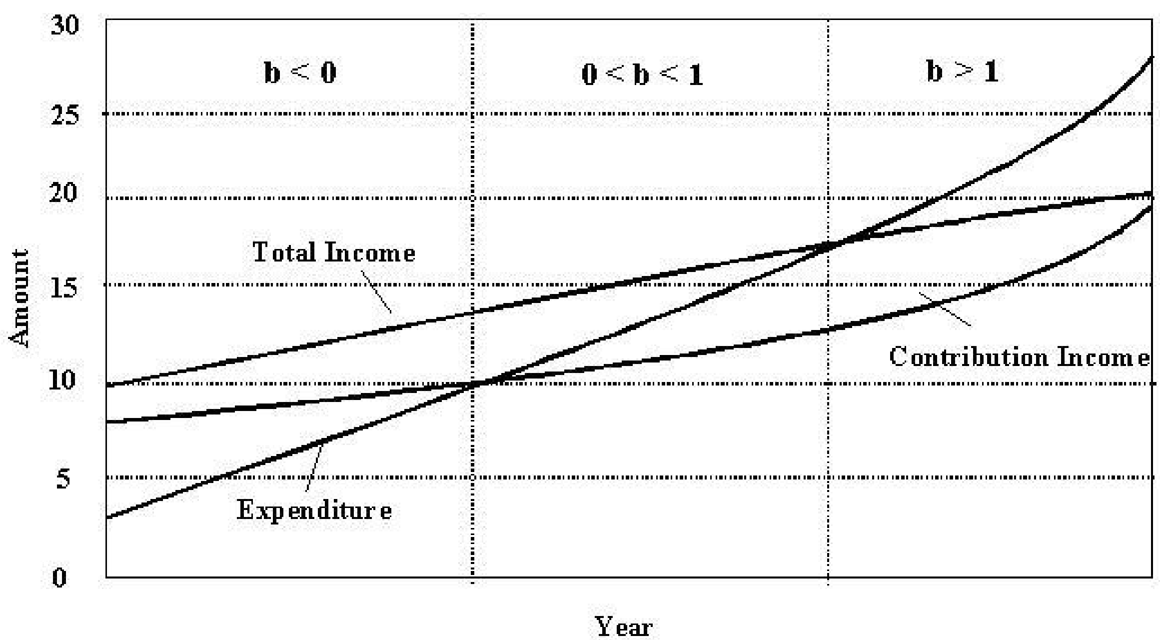
[2] We define the balance ratio only when R_t is strictly positive.

Thus, the balance ratio indicates the percentage of the expected interest income which needs to be liquidated to meet the expenditure.

A significant feature of this financial indicator is that its value characterizes the long-term transition of the financial status of a scheme. This is illustrated as follows:

$b_t < 0$	$\Longleftrightarrow E_t < C_t$	$\Longleftrightarrow$	the contribution income exceeds the expenditure; therefore, there is no need to liquidate the interest income.
$b_t = 0$	$\Longleftrightarrow E_t = C_t$	$\Longleftrightarrow$	the contribution income is just equal to the expenditure; the scheme is in the pay-as-you-go state.
$0 < b_t < 1$	$\Longleftrightarrow C_t < E_t < I_t$	$\Longleftrightarrow$	the contribution income is less than the expenditure; however, if the interest income is taken into account, the balance is positive (in surplus).
$b_t = 1$	$\Longleftrightarrow E_t = I_t$	$\Longleftrightarrow$	the sum of the contribution income and the interest income is equal to the expenditure; the balance is zero.
$b_t > 1$	$\Longleftrightarrow E_t > I_t$ (i.e. $\Delta F_t < 0$)	$\Longleftrightarrow$	the total income is less than the expenditure; the balance is negative (in deficit) and the reserve is declining.

Figure. Balance ratio and the long-term transition of financial status



1.3 Applications of financial indicators in selected countries

In this section, we illustrate how financial indicators are actually used in the financing of existing social security pension schemes. For this purpose, four OECD countries have selected, namely the USA, Germany, Japan and Canada.

Firstly, under the Old-Age, Survivors and Disability Insurance (OASDI) of the United States, benefits are adjusted according to changes in the consumer price index (CPI). However, there is a provision[3] which states that if the reserve ratio, named the "OASDI trust fund ratio", is less than 20.0 % at the beginning of the year, then the cost-of-living adjustment in benefits in that year will be limited to the CPI increase or the wage increase, whichever is the lower.[4]

In addition, the Trustee Board of the OASDI Trust Fund has adopted a short-range test of financial adequacy which requires the "OASDI trust fund ratio" to be at least equal to 1.0 throughout the next ten years.

Secondly, under the Employee's Mandatory Pension Insurance of Germany, the contribution rate is determined every year so that the liquid reserve at the end of the year should be at least equal to one month of estimated annual expenditure excluding the Federal subsidy.[5] This condition can be expressed such that the reserve ratio[6] should be more than one-twelfth, i.e. at $a_t \geqq 0.09$.

Thirdly, according to the 1994 actuarial valuation of the Employee's Pension Insurance of Japan[7], the future contribution rate is determined by the following conditions:

1° The contribution rate is raised every five years[8] at a constant rate.

2° A level contribution rate, called the ultimate contribution rate, is applied after the increase in the costs becomes stationary.

3° The balance is maintained positive every year.

4° A certain amount of reserve is set up for unforeseen economic changes.

Conditions 1° and 2° lead to the following formula for the future contribution rate:

$$p_t = \begin{cases} p_0 + \Delta p([(t-t_0)/5]+1) & (\text{for } t_0 \leqq t \leqq T-1) \\ p_{\max} & (\text{for } t \geqq T), \end{cases}$$

where

[3] Social Security Act §215(i).

[4] In this case, when the "OASDI fund ratio" improves to more than 32.0 %, benefits will be retroactively increased to the level calculated without applying the provision. However, the provision has not come into effect since its introduction in 1983, and is not expected to be implemented for the next several years.

[5] Sozialgesetzbuch VI §158. The law also states that the estimated future contribution rates for the next 15 years should be shown in the annual report on the pension insurance (§154).

[6] Strictly, only the liquid reserves should be taken into account in the calculation of the reserve ratio.

[7] Report of the 1994 actuarial revaluation of the Employee's Pension Insurance and National Pension, prepared by the Actuarial Affairs Division of the Pension Bureau, Ministry of Health and Welfare of Japan.

[8] The Employee's Pension Insurance Act provides that the contribution rate must be re-evaluated when the actuarial valuation is carried out every 5 years.(§84(iv)).

t_0	:	Base year of the valuation.
p_0	:	Contribution rate in the base year.
$p_{\max}$	:	Ultimate contribution rate.
Δp	:	Step of increase in the contribution rate for every 5 years.
T	:	Target year of the ultimate contribution rate (assumed to be 2025).
$[x]$	:	The integer part of x.

Conditions 3° and 4° can be interpreted as $b_t \leqq 1$ and $a_t \geqq a_0$, respectively, where a_0 is the required level of the reserve ratio.

Fourthly, in the 15th actuarial valuation of the Canada Pension Plan[9], the so-called "15-year formula" is applied to determine the contribution rates for the period after 2016[10]. This formula is described as follows:

1° The contribution rate is raised every year at a constant rate.

2° The annual rate of increase of the contribution rate is revised every five years.

3° The rate of increase is determined as the lowest rate of increase such that, if it were applied for the next 15 years, the expected reserves at the end of this period would be at least equal to twice the expenditure in the subsequent year.

From condition 1°, the formula for the contribution rates after 2016 is

$$p_t = p_i + \Delta p_i \cdot (t - t_i) \qquad (\text{for } t = t_i + 1, \ldots,\ t_i + 5;\ t_i = t_0 + 5i,\ i = 0, 1, 2, \ldots),$$

where

t_i	:	Years of i-th contribution review after 2016 (recurring every five years).
p_i	:	Contribution rate in year t_i.
Δp_i	:	Annual rate of increase in the contribution rate for $t_i + 1 \leqq t \leqq t_i + 5$.

For each year of contribution review t_i , the rate of increase applied for the next 5 years, Δp_i , is determined in accordance with the condition 3^o, i.e. $a_{t_i+15} \geqq 2$.[11] To estimate a_{t_i+15}, it has been assumed that the contribution rate is raised continuously at the constant annual rate Δp_i until $t = t_i + 15$.

The above-explained provision concerning the schedule of future contribution rates has been changed in the subsequent amendments of the Act. According to the reports of 16th and 17th actuarial valuations[12], the following methods have been applied to set the long-term contribution rates[13].

[9] Canada Pension Plan Fifteenth Actuarial Report as at 31 December 1993, prepared by the Office of the Superintendent of Financial Institutions, pursuant to §115(3) of the Canada Pension Plan Act.

[10] In the 15th actuarial report the annual rates of increase in the contribution rate from 1992 to 2015 were set as follows: 0.2% for 1992-1996, 0.25% for 1997-2006, 0.2% for 2007-2016. This contribution schedule was called the "25-year schedule" and was assumed to be revised every 5 years.

[11] In our notation, the definition of the reserve ratio should be modified as: $a_t = F_t/E_{t+1}$.

[12] Canada Pension Plan Sixteenth Actuarial Report, September 1997. Canada Pension Plan Seventeenth Actuarial Report as at 31 December 1997, prepared by the Office of the Superintendent of Financial Institutions, pursuant to §115(3) of the Canada Pension Plan Act.

[13] It is said that this amendment of contribution schedule is aimed at averting the continuous increase in contribution rates over a long period and achieving a more equal cost sharing among different generations.

1° The schedule of increase in contribution rates until 2002 is prescribed and no subsequent increase is scheduled.

2° For the period in and after 2003, a level contribution rate, called the "steady-state contribution rate" is applied.

3° The "steady-state contribution rate" is determined as the lowest constant rate which will enable for the reserve ratio to remain generally constant.

In condition 3° , the interpretation of the requirement of "generally constant" reserve ratio is left to the judgement of actuaries and may be redetermined for each valuation. In fact, the comparison of the reserve ratios in 2030 and 2100 was adopted (i.e. $a_{2030} = a_{2100}$) in the 16th valuation. In the 17th valuation, however, the reserve ratios in 2010 and 2060 were chosen for this purpose (i.e. $a_{2010} = a_{2060}$. The projection results show that $a_{2010} = 4.12$ and $a_{2060} = 5.38$ under the standard set of assumptions.).

2. Generalized Scaled Premium

2.1 The Basic Problem

In actuarial valuations of social security pensions, the future contribution rates which adequately ensure the long-term solvency of the schemes should be established in view of the projected expenditure and contributory earnings. Experiences in the United States, Germany, Japan and Canada suggest that required conditions in determining the contribution rate are stated in terms of the financial indicators, in particular, the reserve ratio and the balance ratio. To develop methods of financing systems reflecting this point, we formulate the basic problem in the following.

Assume that the initial reserve, assumed future interest rates, and estimated figures of future expenditure and total contributory earnings are given. Then the problem is stated as follows: *if a period of equilibrium and target values of financial indicators are given, then one must find the level contribution rate such that during the period of equilibrium the resulting financial indicators are sufficient to meet their target values.* Clearly, the higher the contribution rate is, the more likely the above condition is met. Therefore, one must find *the lowest possible* contribution rate.

Let $n \leqq t \leqq m$ be the period of equilibrium, and a_0, b_0 , the target values of the reserve ratio and the balance ratio respectively. Let us denote by $GSP_{n,m}(a \geqq a0, b \leqq b0)$ the lowest contribution rate which satisfies the above conditions, and call it the Generalized Scaled Premium with respect to the period of equilibrium $n \leqq t \leqq m$ and target values a_0, b_0. The term "Generalized Scaled Premium" reflects the fact that it provides a generalization of the Scaled Premium, which is the level contribution rate preserving the accumulated reserves during a given period. In fact, if we denote by $SP_{n,m}$ the Scaled Premium for the same period as above, then $SP_{n,m} = GSP_{n,m}(b \leqq 1)$.

2.2 Modification to the Problem

In this section, we set out the procedure to determine the Generalized Scaled Premium. For simplicity, we assume $1 \leqq t \leqq m$ and put $GSP_m(a \geqq a_0, b \leqq b_0)$ for $GSP_{1,m}(a \geqq a_0, b \leqq b_0)$.

In order to calculate $GSP_m(a \geqq a_0)$, we define the contribution rate such that the reserve ratio attains its target value in year $t = d$ $(1 \leqq d \leqq m)$, i.e. $a_d = a_0$, and denote it by $Q_d(a = a_0)$.

Then it follows that[14]

$$GSP_m(a \geqq a_0) = \max\{Q_d(a = a_0)\ ;\ a \leqq d \leqq m\}. \tag{11}$$

In fact, if the above contribution rate is lower than one of the values of $\{Q_d(a = a_0);\ d = 1, .., m\}$, say $Q_h(a = a_0)$ $(1 \leqq h \leqq m)$, then it follows that $a_h < a_0$, which means that the condition does not hold in year $t = h$.

Similarly, we define $Q_d(b = b_0)$ by replacing the reserve ratio by the balance ratio in the definition of $Q_d(a = a_0)$. Then, we have

$$GSP_m\ (b \leqq b_0) = \max\{Q_d(b = b_0)\ ;\ 1 \leqq d \leqq m\}. \tag{12}$$

Finally, we obtain

$$GSP_m(a \geqq a_0, b \leqq b_0) = \max\{GSP_m(a \geqq a_0), GSP_m(b \leqq b_0)\}. \tag{13}$$

The above discussion indicates the following procedure to calculate the Generalized Scaled Premium:

(a) Calculate $Q_d(a = a_0)$ and $Q_d(b = b_0)$ for each year of the given period of equilibrium.

(b) Find the maximum value of them. The result gives the required contribution rate.

In practice, experience has shown that Q_d is a monotonically increasing function of d if the PAYG cost rate increases substantially. In such cases, one needs to calculate only the value at the end year of the period of equilibrium, Q_m.

2.3 Premium Formulae

We develop the formulae for $Q_d(a = a_0)$ and $Q_d(b = b_0)$.

The given data are summarised as follows:

- Interest rate: i.

[14] Strictly, $Q_d(a = a_0)$ makes no sense when $d = 1$.

- Period of equilibrium: $1 \leqq t \leqq m$.
- Target values of the reserve ratio and the balance ratio: a_0, b_0.
- Initial reserve: F_0.
- Estimated values of expenditure and total contributory earnings: $E_t, S_t (1 \leqq t \leqq m)$.

The basic equation of the evolution of the reserve (equation (5)) is rewritten as:

$$vF_t = F_{t-1} + v^{\frac{1}{2}}(pS_t - E_t), \tag{14}$$

where $v = (1+i)^{-1}$ is the discount rate. As we are looking for a level contribution rate, the contribution rate on the right-hand side of (14) is assumed to be independent of year t.

The solution of this recursion formula is

$$F_t = v^{-t} \left(p \sum_{k=1}^{t} v^{k-\frac{1}{2}} S_k - \sum_{k=1}^{t} v^{k-\frac{1}{2}} E_k + F_0 \right) \tag{15}$$

(for $t = 0, 1, 2, \ldots$).[15]

(i) Formula for $Q_d(a = a_0)$

The condition is $a_d = a_0$. Thus, from (9),

$$F_{d-1} = a_0 E_d. \tag{16}$$

Substituting (15) into (16), we obtain the following formula:

$$p = Q_d(a = a_0) = \frac{a_0 v^{d-1} E_d - F_0 + \sum_{k=1}^{d-1} v^{k-\frac{1}{2}} E_k}{\sum_{k=1}^{d-1} v^{k-\frac{1}{2}} S_k} \tag{17}$$

(for $d = 2, 3, 4, \ldots, m$).

Putting $a_0 = 0$ in (17) yields the formula of the Level Premium for the period $1 \leqq t \leqq d-1$. Therefore, the concept of Generalized Scaled Premium also includes the Level Premium as a special case.

(ii) Formula for $Q_d(b = b_0)$

The condition is $b_d = b_0$. Thus, from (10),

$$C_d - E_d + b_0 R_d = 0. \tag{18}$$

Substituting (2) and (4) into (18), we obtain

$$pS_d - E_d + b_0 \left[(\sqrt{1+i} - 1)(pS_d - E_d) + iF_{d-1} \right] = 0. \tag{19}$$

[15] In the case $t = 0$, the summation terms should be interpreted as 0.

By combining (15) and (19), we obtain the following formula:

$$p = Q_d(b = b_0) = \frac{\left(1 + b_0(v^{-\frac{1}{2}} - 1)\right) v^d E_d + b_0(1 - v)\left(\sum_{k=1}^{d-1} v^{k-\frac{1}{2}} E_k - F_0\right)}{\left(1 + b_0(v^{-\frac{1}{2}} - 1)\right) v^d S_d + b_0(1 - v)\sum_{k=1}^{d-1} v^{k-\frac{1}{2}} S_k} \tag{20}$$

(for $d = 1, 2, 3, \ldots, m$).[16]

2.4 Concluding Remarks

The concept of Generalized Scaled Premium is broad enough to cover most known financing methods for social security pensions. The author believes that the Generalized Scaled Premium provides useful information in determining future contribution rates. This actuarial tool enables financial planners to take into account the requirements of the funding level and the cash-flow balance in the financial management of the scheme.

Mathematically, the Generalized Scaled Premium is obtained by putting additional conditions to the basic equation of the evolution of the reserve. This paper does not address the issue on how to determine the adequate required levels of the reserve ratio and the balance ratio. As shown in Section 1.3, each country adopts different requirements and develops its own rule to maintain the long-term solvency of the national social security pension scheme. The author believes that it is in this respect that social security actuaries should provide professional expertise and play an important role in the area of policy-making.

[16] In the case $d = 1$, the summation terms should be interpreted as 0.

Appendix 1: Premium Formulae Under Varying Interest Rates

1. The purpose of this additional note is to generalize the results obtained in the paper with a view to:

   - Consider the situation where the assumed interest rate is time dependent (i.e. $i = i_t$).
   - Give the formula over an arbitrary period of equilibrium $n \leqq t \leqq m$ $(n < m)$.
   - Rewrite the formula in a form which is suited for calculation.

2. When the assumed interest rate depends on time, the basic equation (equation (5) or (14)) is written by[17]

$$F_t = (1 + i_t)F_{t-1} + \sqrt{1 + i_t} \cdot (pS_t - E_t),$$

or

$$v_t F_t = F_{t-1} + v_t^{\frac{1}{2}} \cdot (pS_t - E_t),$$

where $v_t = \dfrac{1}{1 + i_t}$.

To solve this equation we introduce the following notations:

$$V_t = \prod_{k=1}^{t} v_k \qquad ; \qquad W_t = V_{t-1} \cdot v_t^{\frac{1}{2}}$$

(for $t = 1, 2, 3, \dots$).

Put, as a convention, that $V_0 = 1$.

By multiplying V_{t-1} to both sides of the second equation, we have

$$V_t F_t = V_{t-1} F_{t-1} + pW_t S_t - W_t E_t$$

(Note that $V_t = V_{t-1} \cdot v_t$).

From this, for $t \geqq n$,

$$\sum_{k=n}^{t} (V_k F_k - V_{k-1} F_{k-1}) = \sum_{k=n}^{t} (pW_k S_k - W_k E_k)\,.$$

Therefore, we have the solution in the following form:

$$V_t F_t = V_{n-1} F_{n-1} + p(\overline{S}_t - \overline{S}_{n-1}) - (\overline{E}_t - \overline{E}_{n-1}).$$

Here we put

$$\overline{S}_t = \sum_{k=1}^{t} W_k S_k \qquad ; \qquad \overline{E}_t = \sum_{k=1}^{t} W_k E_k.$$

[17] In this note, the same notation is used as defined in the paper, unless defined here.

3. We give the formulae for $Q_d(a = a_0)$ and $Q_d(b = b_0)$ over the period of equilibrium $n \leqq t \leqq m$. We denote these rates by $Q_d(a = a_0; n, m)$ and $Q_d(b = b_0; n, m)$, respectively.

   The given data are summarised as follows:

   - Interest rate: $i = i_t$.
   - Period of equilibrium: $n \leqq t \leqq m$.
   - Target values of the reserve ratio and the balance ratio: a_0 and b_0.
   - Initial reserve: F_{n-1}.
   - Forecasts of the values of expenditure and total contributory earnings: E_t and S_t.

   By similar calculation, corresponding to formula (17) in the paper we have

$$Q_d(a = a_0; n, m) = \frac{a_0 V_{d-1} E_d - V_{n-1} F_{n-1} + (\overline{E}_{d-1} - \overline{E}_{n-1})}{\overline{S}_{d-1} - \overline{S}_{n-1}}$$

(for $d = n + 1, \ldots, m$).

   Corresponding to formula (20) in the paper, we have

$$Q_d(b = b_0; n, m) = \frac{[1 + b_0(v_d^{-\frac{1}{2}} - 1)]V_d E_d + b_0(1 - v_d)(\overline{E}_{d-1} - \overline{E}_{n-1} - V_{n-1} F_{n-1})}{[1 + b_0(v_d^{-\frac{1}{2}} - 1)]V_d S_d + b_0(1 - v_d)(\overline{S}_{d-1} - \overline{S}_{n-1})}$$

(for $d = n, \ldots, m$).

4. We derive a more general formula for $Q_d(b = b_0)$.

   Recall the short-term liquidity condition:

$$C(t) + H(t) \geqq E(t),$$

   where

$C(t)$	:	Contribution income (in cash) in year t
$H(t)$	:	Cash income other than contributions in year t
$E(t)$	:	Expenditure in year t.

   A theoretical range of $H(t)$ is given by:

$$0 \leqq H(t) \leqq F(t - 1) + I(t),$$

   where

$F(t-1)$	:	Reserve at the beginning of year t
$I(t)$	:	Investment income in year t.

   Thus,

$$H(t) = \alpha I(t) + \beta F(t - 1),$$

   with $\alpha, \beta \in [0, 1]$.

Here, α and β represent the portion of liquidable investment income and that of reserves, respectively.

We define the *liquid ratio* by

$$l(t) = \frac{E(t) - C(t)}{H(t)} = \frac{E(t) - C(t)}{\alpha I(t) + \beta F(t-1)}.$$

In the case where the expenditure exceeds the contributions income, the liquid ratio represents the excess expenditure as a percentage of the total liquidable income. Note that $l(t) = b(t)$ if $\alpha = 1$ and $\beta = 0$.

Consider the GSP that guarantees a given target value of l_0 for a period $[n, m]$ (for given α and β). Then by similar process, we have

$$GSP_{n,m}(l \geqq l_0) = \max\{Q_d(l = l_0)\ ;\ n \leqq d \leqq m\}$$

and

$$Q_d(l = l_0; n, m) = \frac{[1 + l_0\alpha(v_d^{-\frac{1}{2}} - 1)]V_d E_d + l_0[\alpha(1 - v_d) + \beta v_d](\overline{E}_{d-1} - \overline{E}_{n-1} - V_{n-1}F_{n-1})}{[1 + l_0\alpha(v_d^{-\frac{1}{2}} - 1)]V_d S_d + l_0[\alpha(1 - v_d) + \beta v_d](\overline{S}_{d-1} - \overline{S}_{n-1})}$$

$$(\text{for } d = n, \ldots, m).$$

Note that if $\alpha = 1$ and $\beta = 0$, we have $Q_d(l = l_0) = Q_d(b = l_0)$.

Appendix 2: Notes on the Long-Term Solvency and Financial Management

The paper involves certain simplified assumptions in order to focus on theoretical aspects of the financing. This appendix discusses about the financial management of actual schemes and summarises the points to be taken into account to ensure the long-term solvency of the schemes.

Cash flow and solvency of a social security pension scheme

Suppose that the benefit structure of a pension scheme is defined, the scheme then has to finance the promised payment in full by every payment period. A pension scheme is called solvent if it can adequately achieve this obligation in the long-term.

To meet its costs, the scheme receives income. Three types of income are available; first, contributions collected from workers and employers are the primary income to the scheme, second, some pension schemes -usually universal coverage schemes- receive subsidies from the State Treasury to meet part of their expenditure; third, if the scheme retains assets as a result of past surplus, investment income is also expected.

As the scheme is required to make expenditure in cash, the liquidity (by this term, we mean the possibility to convert into cash) of these types of income is particularly important when we examine the financial solvency of the scheme. Even if the scheme retains a large amount of reserves, it is not solvent unless these reserves or their interest are readily realised when needed to meet expenditure.

In terms of liquidity, contributions are collected from workers' payroll and are thus equivalent to cash. The state subsidy comes from general tax revenue therefore is classified as cash unless it is paid in the form of loans or bonds. Contrary to these, interest or dividends are generally credited on the assets in investment and usually special arrangements are necessary to liquidate them.

In relation to this, it should be noted that there is usually a certain time lag between the collection of contributions and their receipt due to administrative procedures. Therefore, pension schemes usually keep a certain amount of liquid assets (such as currency or bank deposits in current accounts) as working capital to enable smooth payment in case of unforeseen delays in receiving expected income. Keeping constantly liquid assets at least equal to the amount expected for the next payment date is considered enough for this purpose.

Typical financial assets available to public pension schemes include (national and corporation) bonds, debentures, loans, stocks, shares and so forth. A pension scheme may also retain non-financial assets such as land, office buildings or public halls.

Characteristics of assets in investment

These various investment instruments can be examined from the following criteria:

In terms of certainty (or stability) of assets, securities like bonds have advantages to others as they give the holders the unconditional right to receive fixed interest on specified dates and to fixed sums as repayments of principals on a specified date. On the other hand, the other assets do not provide the right to a pre-determined income and their values are inevitably subject to the market value fluctuation. It should be noted, however, that no financial asset can avoid the risk of default and that of loss of real value due to inflation.

Yield (interest or dividend) is a trade-off factor for safety. In general, investment which expects high return may involve high risk, and vice versa. Achieving a high return of investment can reduce or postpone the need to raise the contribution rate. However, for public pension schemes the safety is the foremost important requirement with respect to any other criteria; therefore, prudent investment regulations should not be sacrificed by taking high risk investments. Nevertheless, it should be sought to at least maintain the real value of the reserve, and further, to achieve an adequate positive real rate of return from adequately profitable investments at reasonable risk. Yield is also a complementary factor of liquidity. For instance, long-term bonds generally provide higher rate of interest than short-term bonds. (In a liberalised market, the opposite situation sometimes occurs.) Currency bears no interest.

Marketability (salability) of assets refers to the ease with which assets can be traded in the market. This is directly related to the development of a (domestic) financial market (in particular, after market). Given the considerable size of pension schemes in national economies, when the scheme needs to dispose its assets (bonds before maturity date or stocks or real estate), if the market is not capable to absorb them or levies high transaction costs, the required cash-inflow is not assured.

Relation between the Government budget and social security funds

In many countries, treasury bonds comprise significant proportions of pension schemes' investment portfolios. This is ascribed to the fact that treasury bonds are considered to be safe since the principal is protected and the interest is guaranteed by the Government. (Some countries make it obligatory for pension schemes to hold a certain level of treasury bonds). Further, in the absence of appropriate alternative investment options due to not fully developed (domestic) financial markets (in particular, issue markets), the schemes may de facto become large holders of treasury bonds. (This concern should be taken into account in particular when setting the contribution rate for a newly implemented scheme).

The credit of Government securities is ultimately backed by the State's power to tax. However, experience in some countries has shown that the surplus of public pension schemes is de facto used to cover the Government's deficit and the substantial part of the reserves of pension schemes simply represents hidden public debt. In that case, if the Government fails to collect the necessary tax or to allocate the necessary budget to redeem the treasury bonds when they need to be liquidated, the scheme may face a liquidity crunch. (Here we have excluded the possibility for the Government to print money to redeem the bonds). Even if the financial status of the Government is not unfavourable, most treasury bonds held by pension schemes are de facto treated as perpetual bonds in the sense that most matured bonds are used to repurchase newly issued bonds.

Social security contributions are collected for the specific purpose of providing benefits to the insured workers. Therefore, the use of social security contributions for other activities of the State (which should, in principle, be financed by current tax income) would be acceptable only if they contribute to the growth of the economy and, as a result, increase the future tax base and contributory base (and further, to improve the quality of life of all nationals as these improvements can be regarded as social welfare in broad terms). Due to its inherent speculative nature, there is nothing certain about the a priori assessment of its risk and chance, except the general fact that a social security scheme forms a substantial component of the national economy therefore the socio-economic and political situation may inevitably have an influence on it.

Long-term financial management and actuarial valuations

In actuarial valuations, the financial operations of the scheme are estimated on the basis of projected values of expenditure and the insurable base. These results provide important information for financial managers in order to take into account the cash-flow requirements.

In view of the fact that typical long-term projections of pension benefits show a gradual increase in the costs over a long period of time, the points which financial managers should bear in mind according to the cash-flow situation of the scheme can be summarised as follows.

In case no negative primary cash-flow balance is estimated in the future (i.e. contributions exceed expenditure), there is no liquidity shortage so far as contributions are regularly received. In this phase, the focus by financial planners should be put on the development of an investment strategy for an optimal portfolio selection from various choices of risk and returns.

The reserve comprises of liquidable assets and non-liquidable assets, and the liquidability of assets may change according to time. For every period, the possible financial operations to be accomplished by financial managers are to re-allocate or to roll-over the assets which become newly liquidable during that period. Possible liquidable assets include cash surplus, liquidable interest (e.g. coupons) and dividends, matured bonds, and disposable assets. (This cash-flow matching problem can be mathematically formulated in the language of stochastic process with respect to the composition of reserves by kind, maturity term, yield and liquidity).

In case a negative primary cash-flow is foreseen (i.e. cash income is not enough to cover the expenditure), if the benefit structure and the contribution rate remain unchanged, the expenditure in excess of contributions has to be met by other revenues.

In view of the large share and low salability of national bonds, a possible method which does not require rapid changes is to choose the dates of maturity of the bonds in anticipation of the projected cash-deficit and redeem them as they need to be used. Since the implementation of this policy entails some effects on the Government budget, an efficient administrative coordination is required between the social security organization and the Treasury.

Experience in countries facing population ageing has shown that once cash-flow turns negative it tends to persist and its magnitude tends to grow rapidly. The year of exhaustion of the reserve, estimated by actuarial projections, is a result of the simulation assuming that the reserve can be fully liquidated as it is needed to meet expenditure. In fact, not all assets are liquidable for any scheme. Therefore, the scheme will be confronted with an insolvency problem due to liquidity shortage before the estimated year of exhaustion.

In the event of insolvency of the social security fund, the failure to meet benefit payments may cause serious social disorder. In order to maintain public confidence in the scheme, a default provision is adopted in many countries which requires the Government's intervention as the guarantor of last resort. However, Government intervention is justifiable only to allow time for cost containment measures to restore the solvency of the scheme. There is no reason (nor capacity) for the Government to take an unlimited liability of the scheme which does not attain financial equilibrium in the long-term. (If the scheme does not cover all nationals, it may also result in income transfer from the non-covered population to the covered population).

Cost containment measures consist in increasing income and decreasing expenditure. Based on the diagnosis of the current scheme, several policy packages can be formulated from various options of measures on both income and expenditure sides, and presented with their financial implication to policy makers for decision. Extensive reforms require a sufficiently long transition period for implementation, since abrupt policy changes may impose sudden changes in the lifetime plans of workers near retirement age. With a view to the long-term nature of pension reform, the financial status of the scheme should be checked on a regular basis and preemptive measures should be taken to ensure the long-term viability of the scheme.

Appendix 3: Notes on Interest Rate

This appendix summarises some basic facts concerning the interest rate.

1. Internal rate of return of a financial asset

Suppose a financial asset which entitles n future payments, denoted by $s_1, ..., s_n$. Assume the payment s_k is due at time t_k (for $k = 1, 2, ..., n$). If one purchases this asset at a price p, then the rate of return can be calculated in the following methods.

(i) Simple interest (payments bear no interest)

The condition is

$$(1 + i\tau_n) \cdot p = \sum_{k=1}^{n} s_k \ ; \ \text{hence, } i = \frac{\sum_{k=1}^{n} s_k - p}{\tau_n p}.$$

(ii) Simple interest (payments bear simple interest)

The condition is

$$(1 + i\tau_n) \cdot p = \sum_{k=1}^{n} s_k(1 + i(\tau_n - \tau_k)) \ ; \text{hence, } i = \frac{\sum_{k=1}^{n} s_k - p}{\tau_n p - \sum_{k=1}^{n-1} s_k \cdot (\tau_n - \tau_k)}.$$

(iii) Compound interest (payments bear compound interest)

The condition is

$$(1 + i)^{\tau_n} p = \sum_{k=1}^{n} s_k \cdot (1 + i)^{\tau_n - \tau_k},$$

or

$$p = \sum_{k=1}^{n} s_k \cdot v^{\tau_k} =: \Phi(v), \ \text{where } v = \frac{1}{1 + i}.$$

The solution to the above equation can be found by standard numerical methods.

(a) Line search (Regula falsi)

Note that $\Phi(v)$ is an increasing function of $v (0 \leqq v \leqq 1)$ and that $\Phi(0) = 0 < p$ and usually $\Phi(1) = \sum s_k > p$. Depending on whether $\Phi(\frac{1}{2}) > p$ or $< p$, the range of the solution to this equation is $[0, \frac{1}{2}]$ or $[\frac{1}{2}, 1]$. Continuing this we can limit the range of the solution.

(b) The Newton-Raphson method

The solution is approximated by the following scheme:

$$v_{i+1} = v_i - \frac{\Phi(v_i) - p}{\Phi'(v_i)} = v_i - \frac{\sum_k s_k v_i^{\tau_k} - p}{\sum_k s_k \tau_k v_i^{\tau_k - 1}}.$$

The initial value may be taken as $v_0 = 1$.

(c) A method using the convexity of Φ

A simpler scheme using the fact that $\Phi(v)$ is a convex increasing function of $v(0 \leqq v \leqq 1)$ is given as follows:

$$v_{j+1} = \frac{\Phi(v_{sol})}{\Phi(v_j)} \cdot v_j = \frac{p}{\sum_k s_k v_j^{\tau_k}} \cdot v_j,$$

where the initial value is $v_0 = 1$.

If the holder sells this asset at price q at time $\tau_s(\tau_{m-1} \leqq \tau_s < \tau_m)$, then the resulting rate of return is calculated by applying the same method for the cash stream $S_k = s_k$ due at τ_k (for $k = 1, 2, ..., m-1$) and $S_m = q$ due at τ_s.

Example

Suppose an investor pays P for the purchase of a security bond with a face amount F, yearly coupons C (payable at the end of the year) and remaining running time n years. In the above notation, the cash stream is written

$$\tau_k = k \text{ (for } k = 1, 2, ..., n); \; s_1 = s_2 = \cdots = s_{n-1} = C, \text{and } s_n = C + F.$$

The formulae corresponding to the cases (i) and (ii) can be written as follows:

Case (i)

$$i = \frac{nC + F - P}{nP} = \frac{C + \frac{F-P}{n}}{P}.$$

Case (ii)

$$i = \frac{nC + F - P}{nP - \frac{n(n-1)}{2}C} = \frac{C + \frac{F-P}{n}}{P - \frac{n-1}{2}C}.$$

Case (iii) (approximation formula using method (c))

$$v_{j+1} = \frac{p}{\sum_{k=1}^{n} C v_j^k + F v_j^n} \cdot v_j.$$

2. Annual rate of return of a financial institution

Suppose a financial institution which deals with cash-flows and retains certain assets as reserves. Consider an operation of a year. Time for that year is denoted by $0 \leqq t \leqq 1$. Let

$F(t)$	:	Reserve at time t.
$C(t)$	:	Cash-flow incurring at time t.
C	:	Total cash-flow during the year $(= \int_0^1 C(t)dt)$.
I	:	Investment income during the year.
A	:	Reserve at the beginning of the year $(= F(0))$.
B	:	Reserve at the end of the year $(= F(1))$.

If there is no other financial flow to the institution, by accounting requirement, $B-A=C+I$ holds as an identity.

The force of return, denoted by d, is defined by

$$dF(t) = \delta F(t)dt + C(t)dt.$$

Hence,

$$F(t) = F(0)e^{\delta t} + \int_0^t C(s)e^{\delta(t-s)}ds.$$

The rate of return is defined by $i = e^{\delta} - 1$.

If $A, B, C(t)$ are known, the value of d is determined by

$$B = Ae^{\delta} + \int_0^1 C(t)e^{\delta(1-t)}dt,$$

and the investment income is given by

$$I = B - A - C = \delta \int_0^1 e^{\delta(1-t)} \left(A + \int_0^t C(s)ds \right) dt.$$

Applying the Newton-Raphson method to find the value of δ in the above equation, we have the following scheme:

$$\delta_{i+1} = \delta_i - \frac{Ae^{\delta_i} + \int_0^1 C(t)e^{\delta_i(1-t)}dt - B}{Ae^{\delta_i} + \int_0^1 C(t)(1-t)e^{\delta_i(1-t)}dt}.$$

Alternatively, if δ is small, the first order approximation will result

$$I = \delta \int_0^1 \left(A + \int_0^1 C(s)ds \right) dt =: \delta W.$$

Therefore,

$$i = e^{\delta} - 1 \simeq \delta = \frac{I}{W}.$$

Note that W means the term-weighted average of the reserves in that year.

Example Suppose that the cash-flow occurs at the middle of the year, i.e. $C(t) = C \cdot \delta(t - \frac{1}{2})$, where $\delta(t)$ is the Dirac delta function (Note that it is different from the force of interest δ used throughout this note). Then,

$$B = Ae^{\delta} + Ce^{\delta/2} = A(1+i) + C\sqrt{1+i}.$$

This approximation has been assumed in the paper.

By solving the above equation with respect to i (assuming linear approximation of the square root, $\sqrt{1+x} = 1 + \frac{1}{2}x$), we have:

$$i = \frac{2I}{A + B - I}.$$

This is so-called “Hardy’s formula”.

Chapter 2

On the Scaled Premium Method

1. Introduction

This paper focuses on the Scaled Premium method. Under this financing method, a contribution rate, called the Scaled Premium, is determined so that the contributions and interest income are adequate to meet the expenditure over a specified period of equilibrium. When the total income is no longer sufficient to cover the expenditure during the period, the contribution rate is raised to a new Scaled Premium for another period of equilibrium starting from that year.

The requirement of non-decreasing reserves was first proposed in Zelenka's paper submitted to the First International Conference of Social Security Actuaries and Statistician, organised by the ISSA/ILO in 1956. The technical basis of the Scaled Premium method was established by Thullen's paper submitted to the Fourth above-mentioned Conference in 1966. Since the time it was introduced, the ILO has recommended this financing method in its technical cooperation activities for developing countries, many of which have adopted this method.

2. Definition of the Scaled Premium Method

Assume that the contribution rate is adjusted every T years (e.g. 10 years, 15 years). Let year $t = n$ be a year of contribution rate review, then the subsequent years of contribution review are $t = n+T, n+2T, n+3T$, and so on. For $i \geqq 1$, we put $T_i = [n+(i-1)T,\, n+iT]$ and call it the i-th period of equilibrium.

Suppose that the amount of reserve at the beginning of year $t = n$ and projected expenditure and insurable base for subsequent years $t \geqq n$ are given. For each period of equilibrium T_i, we define the level contribution rate, called the Scaled Premium SP_i, as the minimum contribution rate such that the reserve should not decline during the period T_i.

More specifically, the procedure is set out as follows : Starting from the initial reserve at the beginning of year $t = n$, the Scaled Premium for the first period SP_1 is determined as the minimum contribution rate that keeps reserves non-decreasing during the period T_1. Starting with the resulting reserve at the end of T_1, the Scaled Premium for the next period T_2 is determined by the same condition. By continuing this procedure, we can set the Scaled Premium SP_i for each subsequent period of equilibrium T_i.

The actual expenditure and contribution are usually different from those projected. If the contribution income based on the above-determined contribution rate and the interest income on the reserve are not sufficient to cover the expenditure in a year $t = N$ in T_1, a new period of equilibrium should be set out and a new Scaled Premium should be calculated starting from $t = N$. Similarly, these premia should be recalculated regularly as the projections of expenditure and contribution base are updated. In general, a Scaled Premium SP_i can preserve the reserve for a period longer than its prescribed period of equilibrium T_i. This extended period is called the maximum period of equilibrium in respect of the contribution rate SP_i.

3. Premium Formula

For $t \geqq n$, put[1]

$S(t)$	:	Contributory earnings;
$B(t)$	:	Benefit expenditure;
$F(t)$	:	Reserve;
$\pi(t)$	:	Contribution rate;
δ	:	Force interest assumed constant over time).

The basic equation describing the evolution of the reserve is as follows:

$$dF(t) = \delta F(t)dt + (\pi(t)S(t) - B(t))dt. \tag{1}$$

This equation implies that the increase in reserve for the period from t to $t+dt$ is attributed to the return on investment of the reserve and the net cash influx to the scheme (i.e. contributions less benefit expenditure) for the same period.

Solving the above equation with respect to $F(t)$, we have

$$F(t) = F(n)e^{\delta(t-n)} + \int_n^t (\pi(u)S(u) - B(u))e^{\delta(t-u)}du. \tag{2}$$

Define $\varphi(t)$ as the contribution rate such that if it is applied for $[n, t]$ then the resulting reserve will take extremum at t, i.e. $F'(t) = 0$. Assuming that the initial reserve $F(n) \geqq 0$ is given, from (2) we have

$$\varphi(t) = \frac{B(t)e^{-\delta t} + \delta \int_n^t B(u)e^{-\delta u}du - \delta F(n)e^{-\delta n}}{S(t)e^{-\delta t} + \delta \int_n^t S(u)e^{-\delta u}du}. \tag{3}$$

[1] These functions are assumed to be differentiable with respect to t.

The Scaled Premium for the period T_1 is determined by

$$SP_1 = \max\{\varphi(t);\ n \leqq t \leqq n+T\}^2.$$

From (2), the resulting reserve is given by

$$F(n+T) = F(n)e^{\delta T} + SP_1 \int_n^{n+T} S(u)e^{\delta(n+T-u)}du - \int_n^{n+T} B(u)e^{\delta(n+T-u)}du.$$

By continuing this process, we can establish Scaled Premium for each subsequent period of equilibrium.

4. Remarks on the Premium Formula

4.1 Interpretation of $\varphi(t)$

When the contribution is set at $\varphi(t)$ for the period $[n, t]$, the reserve at $s \in [n, t]$ is

$$F_{\varphi(t)}(s) = F(n)e^{\delta(s-n)} + \varphi(t) \int_n^s S(u)e^{\delta(s-u)}du - \int_n^s B(u)e^{\delta(s-u)}du,$$

where $\varphi(t)$ is determined by formula (3).

By definition, ${F'}_{\varphi(t)}(t) = 0$. Hence, from (1) we have

$$\varphi(t) = \frac{B(t) - \delta F_{\varphi(t)}(t)}{S(t)} = PAYG(t) - \delta \frac{F_{\varphi(t)}(t)}{S(t)}. \tag{4}$$

This rate is the portion of benefit expenditure in excess of investment income expressed as a percentage of the insurable earnings. We always consider the case in which $\varphi(t) \geqq 0$. (Otherwise, there is no need to collect contributions as the reserve is large enough to cover the expenditure only by its investment income[3]).

4.2 Premium formula for discrete case

In discrete case, formula (3) is approximated by:

$$\varphi(k) \simeq \frac{B_k \cdot \frac{v^{k-\frac{1}{2}}}{1-v} + \sum_{j=n}^{k-1} B_j v^{j-\frac{1}{2}} - F_{n-1}v^{n-1}}{S_k \cdot \frac{v^{k-\frac{1}{2}}}{1-v} + \sum_{j=n}^{k-1} S_j v^{j-\frac{1}{2}}}$$

In the above formula year k is a discrete variable representing year, and the annual amounts S_k and B_k are assumed to occur at the middle of the year $t = k + \frac{1}{2}$ and F_k is evaluated at the end of the year k. We put $v = e^{-\delta}$.

[2] Since $\varphi(t)$ is continuous on the interval $[n, n+T]$, the maximum on the right-hand side always exists (Weierstrass). Therefore, for any period of equilibrium, yhere exists a (unique) Scaled Premium.

[3] This extreme case is called the complete funding.

4.3 A relation between the scaled premium and the level premium

Let

$$LP(n,t) = \frac{\int_n^t B(u)e^{-\delta u}du - F(n)e^{-\delta n}}{\int_n^t S(u)e^{-\delta u}du} \quad ; \quad LP^{\circ}(n,t) = \frac{\int_n^t B(u)e^{-\delta u}du}{\int_n^t S(u)e^{-\delta u}du}.$$

$LP(n,t)$ is the level premium for the period $[n,t]$ taking into account the initial reserve; whereas, $LP^{\circ}(n,t)$ is the level premium for the same period without taking into account the initial reserve. Because $F(n) \geqq 0$, we have $LP^{\circ}(n,t) \geqq LP(n,t)$.

We have

$$\begin{aligned}
\varphi(t) &= \frac{B(t)e^{-\delta t} + \delta \int_n^t B(u)e^{-\delta u}du - \delta F(n)e^{-\delta n}}{S(t)e^{-\delta t} + \delta \int_n^t S(u)e^{-\delta u}du} \\
&= \frac{[S(t)e^{-\delta t}] \cdot PAYG(t) + [\delta \int_n^t S(u)e^{-\delta u}du] \cdot LP(n,t)}{[S(t)e^{-\delta t}] + [\delta \int_n^t S(u)e^{-\delta u}du]} \\
&=: \frac{n(t) \cdot PAYG(t) + m(t) \cdot LP(n,t)}{n(t) + m(t)}.
\end{aligned}$$

Here, $n(t) := S(t)e^{-\delta t} \geqq 0$, $m(t) := \delta \int_n^t S(u)e^{-\delta u}du \geqq 0$. Thus $\varphi(t)$ is written as a weighted average of $PAYG(t)$ and $LP(n,t)$.

Assume that $\delta > S'(t)/S(t)$ i.e. nominal interest rate is greater than nominal rate of wage increase. Then, in the limit $t \to \infty$, we have

$$n(t) = S(t)e^{-\delta t} \to 0 \quad ; \quad m(t) = \delta \int_n^t S(u)e^{-\delta u}du \to \text{bounded}.$$

Hence $\varphi(t) \to LP(n,t)$ (as $t \to \infty$), which means that if the period of equilibrium is taken sufficiently long the Scaled Premium would be close to the level premium.

Similarly, in the limit of $t \to n$, we have

$$n(t) = S(t)e^{-\delta t} \to \text{bounded} \quad ; \quad m(t) = \delta \int_n^t S(u)e^{-\delta u}du \to 0.$$

Hence $\varphi(t) \to PAYG(n) = LP^{\circ}(n,n)$ (as $t \to n$), which means that if the period of equilibrium is taken short the Scaled Premium would be close to the PAYG contribution rate.

5. A Simplified Premium Formula Under Special Conditions

The original definition of the Scaled Premium by Zelenka-Thullen was slightly different from the one made earlier in this paper. In the original definition, the requirement was replaced by the following condition:

"The contribution rate is determined by requiring the reserve to reach a local maximum at the end of the period of equilibrium."

It is seen that the original definition is a special case of our definition. Therefore, if we denote the above defined premium by SP_i°, then $SP_i^\circ \leqq SP_i$.

Under the original definition, the procedure of setting the Scaled Premium is given by the following algorithm.

$$SP_i^\circ = \varphi(n+iT) = \frac{\beta_i + \delta\nu_i - \gamma_{i-1}}{\alpha_i + \delta\mu_i} \quad ; \quad \gamma_i = \gamma_{i-1} + SP_i^\circ \cdot \mu_i - \nu_i \quad (\text{for } i = 1, 2, 3, ...),$$

where

$$\alpha_i = S(n+iT)e^{-\delta(n+iT)} \quad ; \quad \beta_i = B(n+iT)e^{-\delta(n+iT)} \quad ; \quad \gamma_i = F(n+iT)e^{-\delta(n+iT)};$$

$$\mu_i = \int_{n+(i-1)T}^{n+iT} S(u)e^{-\delta u}du \quad ; \quad \nu_i = \int_{n+(i-1)T}^{n+iT} B(u)e^{-\delta u}du.$$

It is natural to ask when these two definitions give the same result starting with the same initial reserve. For this question, we have the following two results, due to Sakamoto and Iyer, which provide sufficient conditions for $\varphi(t)$ to be non-decreasing. It should be noted that the non-decreasing property of $\varphi(t)$ entails that $SP_i^\circ (= SP_i)$ increases as i increases and therefore each period of equilibrium is maximal. Roughly, both Theorems assert that the two definitions give identical results during the maturing period of a scheme (the meaning of this will be made clear in the statement of Theorems), which is a common situation in the early stages of newly implemented social security schemes. We start with a lemma.

Lemma. *If $PAYG'(u) > 0$ on $[n, t]$, then it follows that*
(i) $PAYG(t) > LP^\circ(n, t)$.
(ii) $PAYG(t) \geqq \varphi(t) \geqq LP(n, t)$.
(iii) $F_{\varphi(t)}(t) \geqq 0$.

Proof. (i) Assume not, then $PAYG(u) < LP^\circ(n, t)$ thus $B(u) < LP^\circ(n, t)S(u)$ on $[n, t]$. Taking present value over the period $[n, t]$ will lead to $LP^\circ(n, t) < LP^\circ(n, t)$, a contradiction. (In general, $PAYG(u)$ intersects $LP^\circ(n, t)$ at least once during the period $[n, t]$.)

(ii) Combining (i) and $LP^\circ(n, t) \geqq LP(n, t)$, we obtain $PAYG(t) > LP^\circ(n, t) \geqq LP(n, t)$. From the result in 4.3, it follows that $PAYG(t) \geqq \varphi(t) \geqq LP(n, t)$.

(iii) From $PAYG(t) \geqq \varphi(t)$ and equation (4), we have $F_{\varphi(t)}(t) \geqq 0$. (Q.E.D.)

Theorem 1(Sakamoto). *If $PAYG'(t) \geqq 0$ and $S'(t) \geqq 0$, then $SP^\circ = SP$.*

Proof. We denote the numerator and the denominator on the right-hand side of equation (3) by $P(t)$ and $Q(t)$, respectively. Noting that $P'(t) = B'(t)e^{-\delta t}$ and $Q'(t) = S'(t)e^{-\delta t}$, we have

$$\varphi'(t) = \left(\frac{P(t)}{Q(t)}\right)' = \frac{e^{-\delta t}}{Q(t)^2}\left(B'(t)Q(t) - P(t)S'(t)\right).$$

By simplifying, we have

$$\varphi'(t) = e^{-\delta t}\frac{S}{Q}\left(\frac{B'}{S} - \varphi\frac{S'}{S}\right) = e^{-\delta t}\frac{S}{Q}\left(PAYG' + \frac{S'\cdot\delta F_{\varphi(t)}}{S^2}\right) \qquad \text{(by(4))}.$$

From lemma (iii), $PAYG'(t) \geqq 0$ ensures $F_{\varphi(t)}(t) \geqq 0$. Hence if $PAYG'(t) \geqq 0$ and $S'(t) \geqq 0$, then $\varphi'(t) \geqq 0$. (Q.E.D.)

Remark. $\varphi'(t) \geqq 0 \Leftrightarrow B'(t) \geqq \varphi(t)S'(t)$. What meaning can be assigned to the RHS?

Theorem 2(Iyer). *Under $\varphi(t) \geqq 0$, if $PAYG'(t) \geqq 0$ and $B'(t) \geqq 0$, then $SP^\circ = SP$.*

Proof. It is sufficient to show that $B'Q \geqq PS'$. From the assumption $PAYG'(t) \geqq 0$, it follows that

$$B'S \geqq BS'. \qquad (\alpha)$$

From lemma (ii) and the assumption, $PAYG(t) \geqq \varphi(t) \geqq 0$. Thus,

$$B/S \geqq P/Q \geqq 0. \qquad (\beta)$$

The assumption $B' \geqq 0$ results in $B'S \geqq 0$. In view of this, multiplying each side of (α) and (β) leads to $BB' \geqq BS'P/Q$, thus $B'Q \geqq PS'$. (Q.E.D.)

Remark 1. In general, the statement "$X \geqq Y \geqq 0$ and $Z \geqq W \Longrightarrow XZ \geqq YW$." is false. (A counter example: $X = 2, Y = 1, Z = -2, W = -3$). But if the additional condition $Z \geqq 0$ is imposed it becomes true.

Remark 2. Under $\varphi(t) \geqq 0$ (which is a plausible situation), if Sakamoto's condition holds, then Iyer's condition is automatically met, but not *vice versa*. In fact, $PAYG' \geqq 0$ gives $B'/B \geqq S'/S$. Hence, $S' \geqq 0$ implies $B' \geqq 0$. A marginal example is the case in which $PAYG' \geqq 0$, $S' \leqq 0$ and $B' \geqq 0$ (the total insurable earnings decrease but the expenditure increases). In this case, Theorem 1 does not tell anything but only Theorem 2 implies that $SP^\circ = SP$.

Remark 3. In Theorem 2, if the condition $B'(t) \geqq 0$ holds, $\varphi(n) \geqq 0$ implies $\varphi(t) \geqq 0$ (for $t \geqq n$). In fact, $B'(t) \geqq 0$ leads to $P'(t) = B'(t)e^{-\delta t} \geqq 0$. Hence, $P(t) \geqq P(n) = \varphi(n)Q(n) \geqq 0$.

Remark 4. For an alternative prrof of Theorem 2, see Appendix.

6. Concluding Remarks

The author believes that most basic properties of the Scaled Premium method are covered in this paper. Once the future expenditure and insurable base are projected, one can set out the future contribution rates based on the method explained above.

From an actuarial point of view, it is important to focus on the long-term increase in the contribution rates. The step of the increase in contribution rates, $\Delta SP_i = SP_{i+1} - SP_i$, is therefore a crucial policy parameter in this respect. Although these steps are calculated as a result of contribution setting, for theoretical purposes, simple conditions are desirable which provide information on the steps without undertaking calculation. This problem is still open, in spite of the anticipation that such conditions may be related to further detailed structure (such as concavity or convexity) of PAYG cost rates.

References

Zelenka, A. (1958). "Quelques remarques sur le régime financier", *Actuarial and Statistical Problems of Social Security*, Vol. III, ISSA, Geneva-Rome.

Thullen, P. (1964). "The Scaled Premium System for the Financing of Social Insurance Pension Schemes: Maximum periods of equilibrium", *International Review on Actuarial and Statistical Problems of Social Security*, No. 10, ISSA.

Thullen, P. (1974). *Techniques actuarielles de la sécurité sociale*, ILO, (Chap. VI.3.).

ILO. (1984). *Introduction to Social Security* (3rd ed., Geneva), (Chap. 12.).

Sakamoto, J. (1982). "On the scaled premium method", *Proceedings of Society of Pension Research of Japan*, Vol. 2. (in Japanese).

Iyer, S. N. (1999). *Actuarial Mathematics of Social Security Pensions*, ILO/ISSA.

Appendix

In this appendix, we present the proofs of Theorems 1 and 2 where the force of interest depends on time: $\delta = \delta(t) > 0$.

In this case, the basic equation describing the evolution of the reserve is:

$$dF(t) = \delta(t)F(t)dt + (\pi(t)S(t) - B(t))dt. \tag{1'}$$

Its solution with respect to $F(t)$ is:

$$F(t) = F(n)e^{\int_n^t \delta(s)ds} + \int_n^t (\pi(u)S(u) - B(u))e^{\int_u^t \delta(s)ds}du. \tag{2'}$$

Similarly, we define $\varphi(t)$ by the condition $F'(t) = 0$. Thus we have

$$\varphi(t) = \frac{B(t) + \delta(t)\int_n^t B(u)e^{\int_u^t \delta(s)ds}du - \delta(t)F(n)e^{\int_n^t \delta(s)ds}}{S(t) + \delta(t)\int_n^t S(u)e^{\int_u^t \delta(s)ds}du}. \tag{3'}$$

Two remarks are needed. First, we note that Lemma in the paper holds in this generalized context. In particular, $PAYG(t) \geqq \varphi(t)$.

Second, if we denote $P(t)$ and $Q(t)$ the numerator and the denominator of equation (3'), then we have

$$\begin{aligned}
\varphi'(t) &= \frac{1}{(Q(t))^2}(P'(t)Q(t) - P(t)Q'(t)) \\
&= \frac{1}{Q(t)}\left(B'(t) - S'(t)\varphi(t) - \frac{\delta'(t)}{\delta(t)}(B(t) - S(t)\varphi(t))\right) \\
&= \frac{\delta(t)}{Q(t)}\left[\left(\frac{B(t)}{\delta(t)}\right)' - \varphi(t)\left(\frac{S(t)}{\delta(t)}\right)'\right].
\end{aligned}$$

Theorem 1'. *If $PAYG'(t) \geqq 0$ and $\left(\frac{S(t)}{\delta(t)}\right)' \geqq 0$, then $SP^\circ = SP$.*

Proof. For simplicity we omit the time variable. We have

$$\begin{aligned}
\varphi'(t) &= \frac{\delta}{Q}\left[\left(\frac{B}{\delta}\right)' - \varphi\left(\frac{S}{\delta}\right)'\right] \\
&= \frac{\delta}{Q}\left[\left(\frac{S}{\delta}PAYG\right)' - \varphi\left(\frac{S}{\delta}\right)'\right] \\
&= \frac{\delta}{Q}\left[\left(\frac{S}{\delta}\right)' PAYG + \left(\frac{S}{\delta}\right)PAYG' - \varphi\left(\frac{S}{\delta}\right)'\right] \\
&= \frac{\delta}{Q}\left[\left(\frac{S}{\delta}\right)'(PAYG - \varphi) + \left(\frac{S}{\delta}\right)PAYG'\right] \geqq 0 .
\end{aligned}$$

(Q.E.D.)

Note. This result is due to Akatsuka and Ohnawa. (Report of 1999 Actuarial Valuation of the Employees' Pension Insurance and the National Pension, Ministry of Health, Labour and Welfare, Japan, 2000)

Theorem 2'. *Under* $\varphi(t) \geqq 0$, *if* $PAYG'(t) \geqq 0$ *and* $\left(\frac{B(t)}{\delta(t)}\right)' \geqq 0$, *then* $SP^{\circ} = SP$.

Proof.

$$\begin{aligned}
\varphi'(t) &= \frac{\delta}{Q}\left[\left(\frac{B}{\delta}\right)' - \varphi\left(\frac{S}{\delta}\right)'\right] \\
&= \frac{\delta}{Q}\left[\left(\frac{B}{\delta}\right)' - \varphi\left(\frac{B}{\delta}\frac{1}{PAYG}\right)'\right] \\
&= \frac{\delta}{Q}\left[\left(\frac{B}{\delta}\right)' - \varphi\left(\left(\frac{B}{\delta}\right)'\frac{1}{PAYG} + \left(\frac{B}{\delta}\right)\left(\frac{1}{PAYG}\right)'\right)\right] \\
&= \frac{\delta}{Q}\left[\left(\frac{B}{\delta}\right)'\left(1 - \frac{\varphi}{PAYG}\right) + \varphi\left(\frac{B}{\delta}\right)\frac{PAYG'}{PAYG^2}\right] \geqq 0 \ .
\end{aligned}$$

(Q.E.D.)

Note. This result was communicated by Mr. Ki-Hong Choi. I am thankful for his permission to reproduce his result here.

Chapter 3

A Relation Between the PAYG Contribution Rate and the Entry Age Normal Premium Rate

1. Introduction

In 1966, Aaron stated that "social insurance can increase the welfare of each person if the sum of the rates of growth of population and real wages exceeds the rate of interest" and called this statement the "social insurance paradox" (see the references). The purpose of this note is to clarify the assumptions underlying this statement.

The basic assumptions are as follows:

(i) Consider a defined-benefit pension scheme which provides pensions equal to $f(0 < f < 1)$ times of the last salary and thereafter pensions are indexed in line with the increase in the average wage.

(ii) Assume that all workers join the scheme at age e, and after contributing for n years they all retire at age $r(= e + n)$.

(iii) Assume that all workers earn the same salary S_t in year t, consequently all pensioners receive the same amount of pension $P_t = f \cdot S_t$ in year t.

2. Premium Under Entry Age Normal Financing Method

We calculate the premium rate under the entry age normal method, which is one of the funding methods of financing pension benefits and generally applied for private or occupational pension plans.

Let

i : Nominal rate of interest.
h : Nominal rate of increase of the average salary.
g : Real rate of interest net of salary increase.

Further, we assume

(iv) The above rates are constant over time.

Then we have $(1+i) = (1+g)\cdot(1+h)$. When g and h are small, this is approximated by $i \simeq g+h$, or $g \simeq i-h$.

To calculate this premium, denoted by C^{EAN}, we equate the present values (discounted by the real interest rate) of the contributions and of the benefit for a new entrant in year t:

$$C^{EAN}\cdot S_t \cdot \ddot{a}_{e:n}(g) = P_t \cdot_n |\ddot{a}_e(g).$$

Here, $_c|\ddot{a}_{x:d}(g)$ denotes the c year deferred life annuity at age x payable for d years under the interest rate g.

Thus, we have

$$C^{EAN} = f\cdot\frac{_n|\ddot{a}_e(g)}{\ddot{a}_{e:n}(g)}.$$

Hence,

$$C^{EAN} = f\cdot\frac{\sum_{x=r}^{\omega} l_x v_g^x}{\sum_{x=e}^{r-1} l_x v_g^x}.$$

Here, l_x is the survival function and $v_\delta = (1+\delta)^{-1}$.

3. PAYG Contribution Rate Under Stable Population

We derive the PAYG contribution rate under *stable* population. If constant mortality and fertility rates are applied for a closed population (i.e. no migration), its age distribution will converge to a certain distribution. In addition, the total population will increase (or decrease) at a constant rate. This population is called the stable population, and the corresponding population growth rate is called the intrinsic growth rate.

In our case, the stable population is realized in the following manner.

Further to assumptions (i)-(iv), we assume

(v) The number of new entrants into the scheme grows at a constant rate of p; and, the age-pattern of mortality remains unchanged over time.

We denote by $L(e+k,t)$ the population of age $e+k$ in year t, and by $N(t)$ the number of new entrants in year t, respectively.

Note that $N(t) = N(t-1)\cdot(1+p) = N(0)\cdot(1+p)^t = N(0)\cdot v_p^{-t}$.

By assumption (v),

$$L(e+k,t) = N(t-k)\cdot l_{e+k}/l_e = N(0)\cdot(1+p)^t\cdot(l_{e+k}\cdot v_p^{e+k})/(l_e\cdot v_p^e).$$

From the above expression, it can be seen that the age distribution of the population is time invariant.

The PAYG contribution rate is calculated as follows:

$$C^{PAYG} = \frac{P_t}{S_t}\frac{\sum_{k=n}^{\omega} L(e+k,t)}{\sum_{k=0}^{n-1} L(e+k,t)} = f\cdot\frac{\sum_{x=r}^{\omega} l_x v_p^x}{\sum_{x=e}^{r-1} l_x v_p^x}$$

4. Conclusion

By summarising the results obtained earlier, under assumptions (i)-(v), we have

$$C^{PAYG} = f\cdot\Gamma(p) \text{ and } C^{EAN} = f\cdot\Gamma(g).$$

Here,

$$\Gamma(\delta) = \frac{\sum_{x=r}^{\omega} l_x v_\delta^x}{\sum_{x=e}^{r-1} l_x v_\delta^x}$$

Since $\Gamma(\delta)$ is a decreasing function with respect to δ, we have

Theorem.

(a) $C^{PAYG} < C^{EAN} \iff p > g$; the latter inequality is approximately equivalent to $p+h > i$, i.e. "the sum of population growth and the increase rate of the average wage is higher than the nominal rate of interest".

(b) $C^{PAYG} > C^{EAN} \iff p < g$; the latter inequality is approximately equivalent to $p+h < i$, i.e. "the nominal interest rate is higher than the sum of population growth and the increase rate of the average wage".

In Aaron's paper, similar results were obtained in the case of annuity certain (i.e., no mortality rates were taken into account). The same approach was taken by the World Bank in their policy report. The above "actuarial" approach was taken by Brown, and Ferrara and Drouin (see the references).

It should be noted that mortality rates and fertility rates may change over time, that there could be migration, and that rates of interest and wage increase are also subject to fluctuation. Thus, assumptions (i)-(v) impose considerable restrictions on the demographic and economic condition.

Therefore, the comparison of the PAYG system and the funding system (the entry age normal method is an example of a funding system) on grounds of this theorem needs to be interpreted with care. In order to choose the suitable system, further economic and political considerations should be made that take into account the specific situation of a country.

References

Aaron, H. (1966). "The Social Insurance Paradox", Canadian Journal of Economics, vol. XXIII, No. 3.

World Bank. (1994). *Averting the old-age crisis: Policies to protect the old and promote growth.* Oxford Univ. Press. (Issue Brief 2: Pay-as you-go or Fully funded - Which costs less?)

Ferrara, G., and Drouin, A. (1996). "Observations on actuarial concepts used in a simplified pension model", in *Social Security Financing : Issues and Perspectives*, ISSA.

Brown, R. L. (1995). "Paygo funding stability and intergenerational equity", *Transactions of the Society of Actuaries*, Vol. XLVII. (also SCOR notes February 1995).

United Nations. (1968). *The concept of a stable population.*

Chapter 4

Natural Cubic Spline Interpolation

1. Definition and Existence of Natural Cubic Spline Functions

Definition 1 *Suppose $n+1$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ are given on an interval $[a, b]$, where $a = x_0 < x_1 < \cdots < x_n = b$. A cubic spline interpolant $S(x)$ for these points is a piecewise cubic function that passes these points and continuous up to the second derivative.*

The conditions for $S(x)$ are the following:

(a) $S(x)$ is expressed as a cubic polynomial on each subinterval $[x_j, x_{j+1}]$ (for $j = 0, 1, \ldots, n-1$). (The polynomial may differ with the subinterval.)

(b) $S(x_j) = y_j \qquad (j = 0, 1, \ldots, n)$.

(c) $S_{j+1}(x_{j+1}) = S_j(x_{j+1}) \qquad (j = 0, 1, \ldots, n-2)$.

(d) $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1}) \qquad (j = 0, 1, \ldots, n-2)$.

(e) $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1}) \qquad (j = 0, 1, \ldots, n-2)$.

In addition, if the following boundary conditions are met, S is called natural spline.

(f) $S''(x_0) = S''(x_n) = 0$ (natural boundary condition).

The following Theorem ensures the existence and uniqueness of the natural cubic spline function.

Theorem 1 *In the above situation, there exists a unique natural cubic spline function interpolating $n+1$ data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ on $[a, b]$.*

Proof. [Step 1: Formulation of the problem]

From condition (a), $S_j(x)$ has the following form:

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3 \qquad (j = 0, 1, \ldots, n-1).$$

Put $a_n = S_{n-1}(x_n) = y_n,\ b_n = S'_{n-1}(x_n)$, and $c_n = \frac{1}{2}S''_{n-1}(x_n)$.

From condition (b), it follows

$$S_j(x_j) = a_j = y_j \qquad (j = 0, 1, \dots, n-1). \tag{1}$$

By putting $h_j = x_{j+1} - x_j$ $(j = 0, 1, \dots, n-1)$, condition (c) yields

$$a_{j+1} = a_j + b_j h_j + c_j h_j^2 + d_j h_j^3 \qquad (j = 0, \dots, n-1). \tag{2}$$

The first derivative of $S_j(x)$ is calculated as: $S'_j(x) = b_j + 2c_j(x - x_j) + 3d_j(x - x_j)^2$.

Condition (d) can be written

$$b_{j+1} = b_j + 2c_j h_j + 3d_j h_j^2 \qquad (j = 0, \dots, n-1). \tag{3}$$

The second derivative of $S_j(x)$ is calculated as: $S''_j(x) = 2c_j + 6d_j(x - x_j)$.

Condition (e) can be written

$$c_{j+1} = c_j + 3d_j h_j \qquad (j = 0, \dots, n-2). \tag{4}$$

We will consider the boundary condition (f) later (Step 3).

Therefore, the determination of each $S_j(x)$ is attributed to the following problem:

(*) Given the values of $\{h_j\}_{j=0}^{n-1}$ and $\{y_j\}_{j=0}^{n}$, find the values of $\{a_j, b_j, c_j, d_j\}_{j=0}^{n-1}$ from equations (1)-(4).

[Step 2: Modification to the problem]

From (1),

$$a_j = y_j \qquad (j = 0, 1, \dots, n-1). \tag{1'}$$

From (4),

$$d_j = (c_{j+1} - c_j)/(3h_j) \qquad (j = 0, 1, \dots, n-1). \tag{4'}$$

By substituting (1') and (4') into equations (2) and (3), respectively, we obtain

$$y_{j+1} = y_j + b_j h_j + h_j^2(2c_j + c_{j+1})/3 \qquad (j = 0, 1, \dots, n-1), \tag{2'}$$

and

$$b_{j+1} = b_j + h_j(c_j + c_{j+1}) \qquad (j = 0, 1, \dots, n-1). \tag{3'}$$

By solving (2') in respect of b_j, we have

$$b_j = (y_{j+1} - y_j)/h_j - h_j(2c_j + c_{j+1})/3 \qquad (j = 0, 1, \dots, n-1). \tag{2''}$$

Substituting (2") into equation (3') and replacing $j+1$ by j,

$$h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_j c_{j+1} = 3(y_{j+1} - y_j)/h_j - 3(y_j - y_{j-1})/h_{j-1} \qquad (j = 1, 2, \dots, n-1). \tag{3''}$$

Noting that a_j $(0 \leqq j \leqq n-1)$ are determined by (1'), we remark that

(#) If the values of c_j $(1 \leqq j \leqq n-1)$ are obtained by solving (3") and if c_0 and c_n are specified by the boundary condition (f), then the values of b_j and d_j $(0 \leqq j \leqq n-1)$ can be determined by (2") and (4').

Therefore the problem (*) is attributed to the system of linear equations (3") in respect of unknowns c_j $(0 \leqq j \leqq n)$ with the boundary condition (f).

[Step 3: Reformulation of the modified problem in terms of matrix algebra]

First, from the boundary condition (f), $c_0 = c_n = 0$.

Therefore, equation (3") can be written by the linear system of equations $Ax = b$, where A is the $(n-1) \times (n-1)$ matrix:

$$A = \begin{pmatrix} 2(h_0+h_1) & h_1 & & & \\ h_1 & 2(h_1+h_2) & h_2 & & 0 \\ & h_2 & 2(h_2+h_3) & \cdots & \\ & & h_3 & \cdots & \\ & 0 & & \cdots & h_{n-2} \\ & & & & 2(h_{n-2}+h_{n-1}) \end{pmatrix},$$

and b and x are the $(n-1)$-dimensional vectors:

$$b = \begin{pmatrix} \frac{3}{h_1}(y_2-y_1) - \frac{3}{h_0}(y_1-y_0) \\ \vdots \\ \frac{3}{h_{n-1}}(y_n-y_{n-1}) - \frac{3}{h_{n-2}}(y_{n-1}-y_{n-2}) \end{pmatrix}, \quad x = \begin{pmatrix} c_1 \\ \vdots \\ c_{n-1} \end{pmatrix}.$$

In view of the above remark (#), it is sufficient to show that $Ax = b$ has a unique solution.

[Step 4: Nonsingularity of A]

In general an $(n \times n)$ matrix M is called strictly diagonaly dominant, if

$$|M_{ii}| > \sum_{j=1, j\neq i}^{n} |M_{ij}| \quad (\text{for } i = 1, 2, \ldots, n).$$

Proposition 1 *The above matrix A is strictly diagonally dominant.*

Proof. In fact, noting that $h_i > 0$,

$$\sum_{j=1, j\neq i}^{n-1} |A_{ij}| \leqq h_{i-1} + h_i < 2(h_{i-1} + h_i) = |A_{ii}| \quad (\text{for } i = 1, 2, \ldots, n-1).$$

(Q.E.D. of Proposition 1)

Proposition 2 *If a matrix M is strictly diagonaly dominant, then M is non-singular.*

Proof. To prove this, it is equivalent to show that "$Mx = 0 \Longrightarrow x = 0$".

Assume that $Mx = 0$ has a nonzero solution $x = (x_i)$. Let k be an index such that $0 < |x_k| = \max\{x_j; 1 \leqq j \leqq n\}$. Then,

$$\sum_{j=1}^{n} M_{ij} x_j = 0 \quad (\text{for } i = 1, 2, \ldots, n).$$

In particular, for $i = k$,

$$M_{kk} x_k = - \sum_{j=1, j \neq k}^{n} M_{kj} x_j.$$

Therefore,

$$|M_{kk}||x_k| \leqq \sum_{j=1, j \neq k}^{n} |M_{kj}||x_j|.$$

Hence,

$$|M_{kk}| \leqq \sum_{j=1, j \neq k}^{n} |M_{kj}| \frac{|x_j|}{|x_k|} \leqq \sum_{j=1, j \neq k}^{n} |M_{kj}|.$$

This contradicts the strict diagonal dominance of M. Consequently, the only solution to $Mx = 0$ is $x = 0$, which proves the required statement. (Q.E.D. of Proposition 2)

By virtue of Propositions 1 and 2, the matrix A is nonsingular. Thus, the system $Ax = b$ has a unique set of solutions. This completes the proof of theorem. (Q.E.D. of Theorem 1)

2. Algorithm to Calculate the Coefficients of Spline Functions

In this section, we establish the algorithm to calculate $\{a_j, b_j, c_j, d_j\}_{j=0}^{n-1}$ from the proof of Theorem 1.

First we note that the above-defined matrix A is tridiagonal, i.e. A has the following structure:

$$A = \begin{pmatrix} a_1 & b_1 & & & & \\ c_1 & a_2 & b_2 & & 0 & \\ & c_2 & a_3 & \cdots & & \\ & & c_3 & \cdots & b_{n-2} & \\ & 0 & & \cdots & a_{n-1} & b_{n-1} \\ & & & & c_{n-1} & a_n \end{pmatrix}.$$

In other words, the elements of A outside the diagonal band of width 3 are zero.

Proposition 3 *A nonsingular tridiagonal matrix A can be factored into two simpler matrices: $A = LU$, where*

$$L = \begin{pmatrix} p_1 & & & & & \\ q_1 & p_2 & & & 0 & \\ & q_2 & p_3 & & & \\ & & q_3 & \cdots & & \\ & 0 & & \cdots & p_{n-1} & \\ & & & & q_{n-1} & p_n \end{pmatrix}, \quad U = \begin{pmatrix} 1 & r_1 & & & & \\ & 1 & r_2 & & 0 & \\ & & 1 & \cdots & & \\ & & & \cdots & r_{n-2} & \\ & & 0 & & 1 & r_{n-1} \\ & & & & & 1 \end{pmatrix}.$$

Proof.

$$LU = \begin{pmatrix} p_1 & p_1 r_1 & & & & \\ q_1 & p_2 + q_1 r_1 & p_2 r_2 & & 0 & \\ & q_2 & p_3 + q_2 r_2 & \cdots & & \\ & & q_3 & \cdots & p_{n-2} r_{n-2} & \\ & 0 & & \cdots & p_{n-1} + q_{n-2} r_{n-2} & p_{n-1} r_{n-1} \\ & & & & q_{n-1} & p_n + q_{n-1} r_{n-1} \end{pmatrix}.$$

Hence, if $LU = A$ holds, then its conditions are

$$p_1 = a_1, \qquad p_{j+1} + q_j r_j = a_{j+1} \quad ; \quad p_j r_j = b_j \quad ; \quad q_j = c_j \quad (\text{for } j = 1, \ldots, n-1).$$

This can be solved by recursion:

(1) $p_1 = a_1$.

(2) For $j = 1, ..., n-1$: $\quad p_{j+1} = a_{j+1} - q_j r_j \quad ; \quad r_j = b_j / p_j \quad ; \quad q_j = c_j$.

(Q.E.D. of Proposition 3)

Remark. Note that the nonsingularity of matrix A is guaranteed by Proposition 2. Therefore, the division by p_j is always possible (i.e. every $p_j \neq 0$).

Proposition 4 *Both matrices L and U are nonsingular and the inverse matrices of L and U can be shown explicitly; if we put $L^{-1} = (\Gamma_{ij})$ and $U^{-1} = (\Lambda_{ij})$, then for $1 \leqq i, j \leqq n$,*

$$\begin{cases} \Gamma_{i,i+k} = 0 & (0 \leqq k \leqq n-i)(\text{i.e. } L^{-1} \text{is lower triangular}). \\ \Gamma_{j+k,j} = (-1)^k p_{j+k}^{-1} \displaystyle\prod_{s=0}^{k-1} p_{j+s}^{-1} q_{j+s} & (0 \leqq k \leqq n-j). \end{cases}$$

$$\begin{cases} \Lambda_{j+k,j} = 0 & (0 \leqq k \leqq n-j)(\text{i.e. } U^{-1} \text{is upper triangular}). \\ \Lambda_{i,i+k} = (-1)^k \displaystyle\prod_{s=0}^{k-1} r_{i+s} & (0 \leqq k \leqq n-j) \end{cases}$$

Sketch of proof. The condition $L\Gamma = I_n$ can be written

$$p_1\Gamma_{1,j} = \delta_{1,j}\ ;\ q_i\Gamma_{i,j} + p_{i+1}\Gamma_{i+1,j} = \delta_{i+1,j} \qquad (\text{for } 1 \leqq i \leqq n-1 \text{ and } 1 \leqq j \leqq n).$$

By solving this system of equations by recursion, we have the above formula. The equation $U\Lambda = I_n$ can be solved in a similar way. (Q.E.D. of Proposition 4)

As a consequence, by applying this decomposition to matrix A, the inverse of A can be calculated explicitly: $A^{-1} = (LU)^{-1} = U^{-1}L^{-1}$.

The Excel VBA programme in Appendix provides the algorithm to calculate the coefficients a_j, b_j, c_j and d_j of the spline functions $S_j(x)$.

A typical application of this method is the subdivision of the grouped data. Suppose the size of a population is given for each 5 year age-group. To subdivide these data into single-age (note that for each group the total of the resulting single-age data should reproduce the 5 year age-group data), the following method can give an answer.

(1) Put : $y(0) = 0$, and $y(5k) = pop(0;4) + pop(5;9) + \cdots + pop(5(k-1);(5k-1))$ (for $1 \leqq k \leqq n$).

(2) Find the natural spline interpolant $S(x)$, such that $S(5k) = y(5k)$.

(3) Put : $pop(x;x+1) = S(x+1) - S(x)$ (for $0 \leqq x \leqq 5n-1$).

3. Variational Property of Spline Functions

In this section, we show that spline functions possess certain minimal property from a variational point of view. The purpose of this section is to prove the following assertion.

Theorem 2 *Let S be a cubic spline interpolant of $n+1$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ on $[a, b]$. Then, for every function $f \in C^2[a, b]$ that interpolates these points, i.e. $f(x_i) = y_i (0 \leqq i \leqq n)$, the following inequality holds:*

$$\int_a^b \left(f''(x)\right)^2 dx \geqq \int_a^b \left(S''(x)\right)^2\ dx.$$

Furthermore, the equation holds if and only if $f(x) = S(x)$ on $[a, b]$.

Proof. We start from

$$\int_a^b \left(f''(x) - S''(x)\right)^2 dx = \int_a^b \left(f''(x)\right)^2 dx - \int_a^b \left(S''(x)\right)^2 dx - 2\int_a^b \left(f''(x) - S''(x)\right) \cdot S''(x) dx.$$

First, we shall show that the last term on the right-hand side is equal to zero by virtue of the following lemma.

Proposition 5 *For any* $g \in C^2[a,b]$,

$$\int_a^b g''(x)S''(x)dx = 6\sum_{i=0}^{n} g(x_i)(d_j - d_{j-1}),$$

where

$$S_j(x) = a_j + b_j(x-x_j) + c_j(x-x_j)^2 + d_j(x-x_j)^3 \text{ on } [x_j, x_{j+1}] \qquad (0 \leqq j \leqq n-1),$$

and

$$d_{-1} = d_n = 0.$$

Proof. By integration by parts,

$$I = \int_a^b g''(x)S''(x)dx = [g'(x)S''(x)]_a^b - \int_a^b g'(x)S'''(x)dx = -\int_a^b g'(x)S'''(x)dx.$$

Since ${S'''}_j(x) = 6d_j$ on $[x_j, x_{j+1}]$, we have

$$I = -\sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} g'(x)\cdot 6d_j dx = -6\sum_{j=0}^{n-1} d_j[g(x_{j+1}) - g(x_j)] = 6\sum_{j=0}^{n} g(x_j)(d_j - d_{j-1}).$$

(Q.E.D. of Proposition 5)

By applying the above Proposition for $g(x) = f(x) - S(x)$, we have

$$\int_a^b \left(f''(x) - S''(x)\right) S''(x)dx = 6\sum_{j=0}^{n} \left(f(x_j) - S(x_j)\right)(d_j - d_{j-1}) = 0.$$

Here, we have used $f(x_j) = S(x_j) = y_j$ $(j = 0, \dots, n)$ from the assumption.

Therefore, we have proved

$$\int_a^b \left(f''(x)\right)^2 dx - \int_a^b \left(S''(x)\right)^2 dx = \int_a^b \left(f''(x) - S''(x)\right)^2 dx \geqq 0.$$

From this the inequality follows immediately.

If the equation holds in the above inequality, it follows that

$$g''(x) = f''(x) - S''(x) = 0 \qquad \text{on each subinterval } [x_j, x_{j+1}].$$

Hence, g is a linear function on $[x_j, x_{j+1}]$. Combined with $g(x_j) = g(x_{j+1}) = 0$, it follows that

$$g(x) = f(x) - S(x) = 0 \qquad \text{on each subinterval } [x_j, x_{j+1}].$$

(Q.E.D. of Theorem 2)

Remark 1. The following fact is known (for detail, see Sugihara and Murota in the references). Using the same notation in Theorems 1 and 2, if we put $h = \max\{|x_{j+1} - x_j|; 0 \leqq j \leqq n-1\}$, then

$$\max_{x\in[a,b]} |f(x) - S(x)| \leqq E_0 \max_{x\in[a,b]} |f''(x)|h^2, \qquad \text{where } E_0 = \frac{13}{48}$$

and

$$\max_{x\in[a,b]} |f'(x) - S'(x)| \leqq E_1 \max_{x\in[a,b]} |f''(x)|h, \qquad \text{where } E_1 = \frac{5}{6} + 0.2514976565.$$

Remark 2.

(i) We may assign a physical interpretation to Theorem 2. Consider a piece of fine uniform elastic material (e.g. piano wire). If we fix its both ends and bend it, it will be in equilibrium with a shape which minimizes its bending energy. The bending energy E of this system σ is given

$$E = C \int_\sigma \kappa^2 \, ds,$$

where

C : a constant depending on the material and the moment of inertia of the section,

κ : the curvature of σ,

ds : the line element of σ.

Suppose the shape of the wire is expressed in terms of x-y coordinate: $\sigma : y = f(x)$ $(\alpha \leqq x \leqq \beta)$

$$\kappa = \frac{|y''|}{[\sqrt{1+y'^2}]^3} \ ; \ ds = \sqrt{dx^2 + dy^2} = \sqrt{1+y'^2}dx.$$

Therefore,

$$E = E[f(x)] = C \int_\alpha^\beta \frac{f''(x)^2}{[\sqrt{1+f'(x)^2}]^5} dx.$$

If the shape of the wire is assumed to be approximately parallel to the x-axis, then $y' \simeq 0$. Under this approximation,

$$E = E[f(x)] = C \int_\alpha^\beta [f''(x)]^2 dx.$$

Theorem 2 entails that the cubic spline function is characterized as the unique function (in C^2-class) which minimizes the above integral relating to the bending energy.

Recall the Euler-Lagrange equation for the functional with higher derivatives:

$$\delta J = \Delta \int_a^b L[u, u', u'', \ldots, u^{(n)}]dx = 0 \implies \sum_{i=0}^{n} (-1)^i \left(\frac{d}{dx}\right)^i \frac{\partial L}{\partial u^{(i)}} = 0.$$

Applying this for the above bending energy, $\delta E = 0$ will lead to the condition $S^{(4)}(x) = 0$, which implies that $S(x)$ is a cubic polynomial on each subinterval.

(ii) The curvature of curve σ is defined by

$$\kappa = \left|\frac{d\theta}{ds}\right|,$$

where θ is the angle of the tangent vector.

If this curve is represented by a parameter, i.e. $\sigma = \{(x(t), y(t));\ t \in T\}$ (assume $x'(t)^2 + y'(t)^2 \neq 0$ for any t), then

$$\theta = \arctan \frac{y'(t)}{x'(t)}.$$

Thus,

$$\frac{d\theta}{ds} = \frac{d\theta}{dt} \cdot \frac{dt}{ds} = \frac{\left(\frac{y'(t)}{x'(t)}\right)'}{1 + \left(\frac{y'(t)}{x'(t)}\right)^2} \frac{1}{\sqrt{x'(t)^2 + y'(t)^2}} = \frac{x'(t)y''(t) - x''(t)y'(t)}{x'(t)^2 + y'(t)^2} \frac{1}{\sqrt{x'(t)^2 + y'(t)^2}}.$$

Hence,

$$\kappa = \frac{|x'(t)y''(t) - x''(t)y'(t)|}{[\sqrt{x'(t)^2 + y'(t)^2}]^3}.$$

In particular, in case of $x = t,\ y = f(t)$, we have

$$\kappa = \frac{|y''|}{[\sqrt{1 + y'^2}]^3}.$$

References

Burden, R. L. and Faires, J.D. (1993). *Numerical Analysis* (5th ed.), PWS Publishing Company, Boston.

Sugihara, M. and Murota, K. (1998). *Methods of Numerical Analysis* (in Japanese), Iwanami shoten publisher, Tokyo.

Iri, M. and Fujino, Y. (1985). *Common Practices in Numerical Analysis* (in Japanese), Kyoritsu shuppan, Tokyo.

Fujita, T. (1992). *Graduation* (in Japanese), *Journal of Society of Actuaries of Japan. Suppl.* No. 134.

Appendix: An Excel VBA programme to calculate the coefficients of cubic spline functions

```
Sub Natural_Cubic_Spline()

n = 20           ' Number of nodes

ReDim x(20), Y(20)                              ' input data
'ReDim A(20), B(20), C(20), D(20)               ' ouput data
ReDim h(20), tmp(20), l(20), z(20), u(20)      ' temporary data

For i = 0 To n
x(i) = NODE(i)
Y(i) = TCOVRATE(i)
Next i

For i = 0 To n
A(i) = Y(i)
Next i

For i = 0 To n - 1:   h(i) = x(i + 1) - x(i):   Next i

For i = 1 To n - 1
tmp(i) = 3 / h(i) * (A(i + 1) - A(i)) - 3 / h(i - 1) * (A(i) - A(i - 1))
Next i

l(0) = 1: u(0) = 0: z(0) = 0

For i = 1 To n - 1
l(i) = 2 * (x(i + 1) - x(i - 1)) - h(i - 1) * u(i - 1)
u(i) = h(i) / l(i)
z(i) = (tmp(i) - h(i - 1) * z(i - 1)) / l(i)
Next i

l(n) = 1: z(n) = 0: C(n) = 0

For J = n - 1 To 0 Step -1
C(J) = z(J) - u(J) * C(J + 1)
B(J) = (A(J + 1) - A(J)) / h(J) - h(J) * (C(J + 1) + 2 * C(J)) / 3
D(J) = (C(J + 1) - C(J)) / (3 * h(J))
Next J

'For i = 0 To n - 1
'Cells(i + 2, 4).Value = a(i)
'Cells(i + 2, 5).Value = b(i)
'Cells(i + 2, 6).Value = c(i)
'Cells(i + 2, 7).Value = d(i)
'Next i

End Sub
```

Chapter 5

Sprague Interpolation Formula

1. Derivation of the Formula

Suppose we wish to interpolate between u_0 and u_1 by a polynomial function, provided that six data $u_{-2}, u_{-1}, u_0, u_1, u_2, u_3$ are given. This problem can be geometrically restated that one has to find a curve which interpolates between the points C and D, where data points are denoted by $A(-2, u_{-2}), B(-1, u_{-1}), C(0, u_0), D(1, u_1), E(2, u_2), F(3, u_3)$. In order to determine this polynomial, we impose the following conditions: (by abuse of language, we also denote by p the curve represented by $y = p(x)$ in the x-y plane.)

(a) Find two complementary quartic polynomials f and g such that f passes A, B, C, D and E, and g passes B, C, D, E, and F (note that there are five unknowns and five constraints).

(b) Determine a polynomial p of degree five by the following conditions (there are six unknowns and six constraints):

- The curve p passes both C and D.
- At C, the first and second derivatives of p are equal to those of f, respectively.
- Similarly, at D, the first and second derivatives of p are equal to those of g, respectively.

Let Δ be the forward difference operator: $\Delta u_x = u_{x+1} - u_x$. Then, by Newton's interpolation formula, f is given by : $f_x = (1 + \Delta)^{x+2} u_{-2}$. Hence,

$$f_x = 1 + (x+2)\Delta + \frac{1}{2}(x+2)(x+1)\Delta^2 + \frac{1}{6}(x+2)(x+1)x\Delta^3 + \frac{1}{24}(x+2)(x+1)x(x-1)\Delta^4$$

(for brevity, we put Δ^x for $\Delta^x u_{-2}$).

The first and second derivatives of f are calculated as follows:

$$\frac{df_x}{dx} = \Delta + \frac{1}{2}(2x+3)\Delta^2 + \frac{1}{6}(3x^2 + 6x + 2)\Delta^3 + \frac{1}{24}(4x^3 + 6x^2 - 2x - 2)\Delta^4,$$

and

$$\frac{d^2 f_x}{dx^2} = \Delta^2 + (x+1)\Delta^3 + \frac{1}{12}(6x^2 + 6x - 1)\Delta^4.$$

Hence, at point C,

$$\left[\frac{df_x}{dx}\right]_{x=0} = \Delta + \frac{3}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{12}\Delta^4 = F_1,$$

and

$$\left[\frac{d^2 f_x}{dx^2}\right]_{x=0} = \Delta^2 + \Delta^3 - \frac{1}{12}\Delta^4 = F_2.$$

Similarly, g is given by : $g_x = (1+\Delta)^{x+1}u_{-1}$. Therefore,

$$g_x = 1 + (x+1)\Delta_1 + \frac{1}{2}(x+1)x\Delta_1^2 + \frac{1}{6}(x+1)x(x-1)\Delta_1^3 + \frac{1}{24}(x+1)x(x-1)(x-2)\Delta_1^4$$

(for brevity, we put Δ_1^x for $\Delta^x u_{-1}$).

The first and second derivatives of g are calculated as follows:

$$\frac{dg_x}{dx} = \Delta_1 + \frac{1}{2}(2x+1)\Delta_1^2 + \frac{1}{6}(3x^2-1)\Delta_1^3 + \frac{1}{24}(4x^3 - 6x^2 - 2x + 2)\Delta_1^4,$$

and

$$\frac{d^2 g_x}{dx^2} = \Delta_1^2 + x\Delta_1^3 + \frac{1}{12}(6x^2 - 6x - 1)\Delta_1^4.$$

Hence, at point D,

$$\left[\frac{dg_x}{dx}\right]_{x=1} = \Delta_1 + \frac{3}{2}\Delta_1^2 + \frac{1}{3}\Delta_1^3 - \frac{1}{12}\Delta_1^4,$$

and

$$\left[\frac{d^2 g_x}{dx^2}\right]_{x=1} = \Delta_1^2 + \Delta_1^3 - \frac{1}{12}\Delta_1^4.$$

Since $\Delta^x u_{-1} = \Delta^x(1+\Delta)u_{-2}$, i.e. $\Delta_1^x = \Delta^x(1+\Delta)$, we eliminate Δ_1 from the above expressions. Thus, we have

$$\left[\frac{dg_x}{dx}\right]_{x=1} = \Delta + \frac{5}{2}\Delta^2 + \frac{11}{6}\Delta^3 + \frac{1}{4}\Delta^4 - \frac{1}{12}\Delta^5 = G_1$$

and

$$\left[\frac{d^2 g_x}{dx^2}\right]_{x=1} = \Delta^2 + 2\Delta^3 + \frac{11}{12}\Delta^4 - \frac{1}{12}\Delta^5 = G_2.$$

Assume that

$$p(x) = \alpha x^5 + \beta x^4 + \gamma x^3 + \delta x^2 + \varepsilon x + \varphi.$$

Then,

$$p'(x) = 5\alpha x^4 + 4\beta x^3 + 3\gamma x^2 + 2\delta x + \varepsilon,$$

and

$$p''(x) = 20\alpha x^3 + 12\beta x^2 + 6\gamma x + 2\delta.$$

The constraints of condition (b) are written as follows:

$$\begin{array}{lll}
p(0) & = \varphi & = u_0. \\
p(1) & = \alpha + \beta + \gamma + \delta + \varepsilon + \varphi & = u_1. \\
p'(0) & = \varepsilon & = F_1. \\
p'(1) & = 5\alpha + 4\beta + 3\gamma + 2\delta + \varepsilon & = G_1. \\
p''(0) & = 2\delta & = F_2. \\
p''(1) & = 20\alpha + 12\beta + 6\gamma + 2\delta & = G_2.
\end{array}$$

By solving these simultaneous equations (after lengthy but straightforward calculation), we have

$$p(x) = u_{-2} + (x+2)\Delta u_{-2} + \frac{1}{2}(x+2)(x+1)\Delta^2 u_{-2} + \frac{1}{6}(x+2)(x+1)x\Delta^3 u_{-2}$$
$$+\frac{1}{24}(x+2)(x+1)x(x-1)\Delta^4 u_{-2} + \frac{1}{24}x^3(x-1)(5x-7)\Delta^5 u_{-2}.$$

Put

$$a(x) = x+2; \quad b(x) = \frac{1}{2}(x+2)(x+1); \quad c(x) = \frac{1}{6}(x+2)(x+1)x;$$
$$d(x) = \frac{1}{24}(x+2)(x+1)x(x-1); \quad e(x) = \frac{1}{24}x^3(x-1)(5x-7).$$

Then, the above result is expressed as a linear combination of the data points:

$$(*) \qquad p(x) = eu_3 + (d-5e)u_2 + (c-4d+10e)u_1 + (b-3c+6d-10e)u_0$$
$$+(a-2b+3c-4d+5e)u_{-1} + (-a+b-c+d-e)u_{-2}$$

(for brevity, $a = a(x)$ etc.)

The following expression may be convenient to remember the above formula.

$$p(x) = \begin{pmatrix} a & b & c & d & e \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -3 & 3 & -1 \\ 0 & 1 & -4 & 6 & -4 & 1 \\ 1 & -5 & 10 & -10 & 5 & -1 \end{pmatrix} \begin{pmatrix} u_3 \\ u_2 \\ u_1 \\ u_0 \\ u_{-1} \\ u_{-2} \end{pmatrix}$$

The elements of the above matrix are coefficients appearing in the expansion of $(X-Y)^n$.

Remarks.

From the expression (*), the Sprague formula is called a type of osculatory interpolation formulae. A similar idea can lead to a variant interpolation formula. For example, the King-Karup formula can be derived from the following problem:

"Interpolate between u_0 and u_1 by a polynomial function, provided that four data $A(-1, u_{-1}), B(0, u_0), C(1, u_1), D(2, u_2)$ are given."

Conditions to determine the interpolating polynomial:

(a) Find two complementary quadratic polynomials f and g such that f passes A, B, C, and g passes B, C, D.

(b) Determine a polynomial p of degree three (cubic) by the following conditions:

- The curve p passes both B and C.
- At B, the first and second derivatives of p are equal to those of f, respectively.
- Similarly, at C, the first and second derivatives of p are equal to those of g, respectively.

The result is given as follows:

$$p(x) = u_0 + x\Delta u_{-1} + \frac{1}{2}x(x+1)\Delta^2 u_{-1} + \frac{1}{2}x^2(x-1)\Delta^3 u_{-1}.$$

It should be noted that a similar consideration is necessary as explained in the next section when applying this formula to the end-point case.

2. Consideration at End Points

If the given data are near the end of the data series, one might not be able to find six points, but five points (i.e. u_3 is missing - next-to-end case) or four points (u_2 and u_3 are missing - end-point case).

A way usually used to cope with this problem is to extrapolate the missing data according to certain rules.

In the publication by Shryock and Siegel (see reference), Table C-2 (p. 876) shows the coefficients of the Sprague formula at intervals of 0.2. In this table, the following assumptions have been made concerning the end-point and next-to-end cases:

- In the next-to-end case, define u_3 so that $\Delta^5 u_{-2} = 0$.
- In the end-point case, apply Newton's interpolation formula of degree three to data u_{-2}, u_{-1}, u_0, u_1. (Note that this is a simple polynomial interpolation).

However, in the case of population, which is expected to have a smooth shape around the end points, it would be more natural to assume the linear extrapolation there. Thus, we propose

- In the next-to-end case, put $u_3 = u_2 + (u_2 - u_1) = 2u_2 - u_1$,
- In the end-point case, put $u_2 = u_1 + (u_1 - u_0) = 2u_1 - u_0$, and $u_3 = 2(2u_1 - u_0) - u_1 = 3u_1 - 2u_0$

We shall apply the latter set of assumptions to the examples we discuss in the following part.

The above argument holds true for the case in which data at the other end points (u_{-2} and/or u_{-1}) are missing. In practice, one should put $w_i = u_{1-i}$ (for $i = -2, -1, 0, 1, 2, 3$) and apply the above method.

3. Applications to Subdivision of Grouped Data

Example 1.

Let us consider the interpolation at intervals of 1/5. By substituting $x = 0.0, 0.2, 0.4, 0.6, 0.8$ in formula (*), we obtain the coefficients $\{\rho_j^i\}$ (for $0 \leqq i \leqq 5$ and $-2 \leqq j \leqq 3$), where

$$u(i/5) = \sum_{j=-2}^{3} p_j^i u(j).$$

These coefficients are shown in the table in the Appendix.

Example 2. (Subdivision of 5-year abridged data)

Suppose population data are given for each 5 year age-group. Consider the subdivision of these data into single-age data. (Note that for each group the total of the resulting single-age data should reproduce the original 5 year abridged data).

The following procedure gives an answer using the Sprague interpolation formula:

Let $\phi^{(d)}(x)$ denotes the population aged from x to $x + d - 1$. Then, the given data are: $\phi^{(5)}(5k)$ (for $0 \leqq k \leqq n - 1$).

(1) Put : $\Phi(0) = 0$, and $\Phi(5k) = \sum_{h=0}^{k-1} \phi^{(5)}(5h)$ (for $1 \leqq k \leqq n$).

(2) Interpolate $\Phi(5k)$ at intervals of 1/5. The results are given by using the coefficients $\{\rho_j^i\}$ calculated in Example 1,

(3) For $0 \leqq k \leqq n-1$ and $0 \leqq i \leqq 4$, assume $\Phi(5k+i) = \sum_{j=-2}^{3} \rho_j^i \Phi(5(k+j))$

$$\phi^{(1)}(5k+i) = \Phi(5k+i+1) - \Phi(5k+i) = \sum_{j=-2}^{3} (\rho_j^{i+1} - \rho_j^i)\Phi(5(k+j))$$

$$\begin{gathered} = (\rho_{-2}^{i+1}-\rho_{-2}^i)[\Phi(5k)-\phi^{(5)}(5(k-1))-\phi^{(5)}(5(k-2))]+(\rho_{-1}^{i+1}-\rho_{-1}^i)[\Phi(5k)-\phi^{(5)}(5(k-1))] \\ +(\rho_0^{i+1} - \rho_0^i)\Phi(5k) + (\rho_1^{i+1} - \rho_1^i)[\Phi(5k) + \phi^5(5k)] \\ +(\rho_2^{i+1} - \rho_2^i)[\Phi(5k) + \phi^{(5)}(5k) + \phi^{(5)}(5(k+1))] \\ +(\rho_3^{i+1} - \rho_3^i)[\Phi(5k) + \phi^{(5)}(5k) + \phi^{(5)}(5(k+1)) + \phi^{(5)}(5(k+2))] \end{gathered}$$

$$\begin{gathered} = (-\rho_{-2}^{i+1} + \rho_{-2}^i)\phi^{(5)}(5(k-2)) + (-\rho_{-2}^{i+1} + \rho_{-2}^i - \rho_{-1}^{i+1} + \rho_{-1}^i)\phi^{(5)}(5(k-1)) \\ +(\rho_1^{i+1} - \rho_1^i + \rho_2^{i+1} - \rho_2^i + \rho_3^{i+1} - \rho_3^i)\phi^{(5)}(5k) \\ +(\rho_2^{i+1} - \rho_2^i + \rho_3^{i+1} - \rho_3^i)\phi^{(5)}(5(k+1)) + (\rho_3^{i+1} - \rho_3^i)\phi^{(5)}(5(k+2)). \end{gathered}$$

Therefore, the result is again written as the linear combination of given data, i.e.,

$$\phi^{(1)}(5k+i) = \sum_{j=-2}^{2} \sigma_j^i \phi^{(5)}(5(k+j)).$$

These coefficients $\{\sigma_j^i\}$ (for $1 \leqq i \leqq 4$ and $-2 \leqq j \leqq 2$) can be calculated from $\{\rho_j^i\}$ (for $0 \leqq i \leqq 5$ and $-2 \leqq j \leqq 3$) and are also shown in the Table in the Appendix.

References

Benjamin, B. and Pollard, J.H. (1980). *The Analysis of Mortality and other Actuarial Statistics*, (2nd ed.), Heinemann, London.

Shryock, H.S. and Siegel, J.S. (1973). *The Methods and Materials of Demography*, U.S. Department of Commerce.

Fujita, T. (1992). *Graduation* (in Japanese), *Journal of Society of Actuaries of Japan. Suppl.* No. 134.

Appendix. Table of coefficients of the Sprague interpolation formula

1. Middle panel (normal case)

Interpolation	$u(0)$	$u(0.2)$	$u(0.4)$	$u(0.6)$	$u(0.8)$	$u(1)$
$u(3)$	0.0000	0.0016	0.0080	0.0144	0.0128	0.0000
$u(2)$	0.0000	-0.0256	-0.0736	-0.1136	-0.0976	0.0000
$u(1)$	0.0000	0.1744	0.4384	0.7264	0.9344	1.0000
$u(0)$	1.0000	0.9344	0.7264	0.4384	0.1744	0.0000
$u(-1)$	0.0000	-0.0976	-0.1136	-0.0736	-0.0256	0.0000
$u(-2)$	0.0000	0.0128	0.0144	0.0080	0.0016	0.0000

Subdivision	1st fifth	2nd fifth	3rd fifth	4th fifth	5th fifth
$G(2)$	0.0016	0.0064	0.0064	-0.0016	-0.0128
$G(1)$	-0.0240	-0.0416	-0.0336	0.0144	0.0848
$G(0)$	0.1504	0.2224	0.2544	0.2224	0.1504
$G(-1)$	0.0848	0.0144	-0.0336	-0.0416	-0.0240
$G(-2)$	-0.0128	-0.0016	0.0064	0.0064	0.0016

2. Next-to-end panel ($u(3)/G(2)$ are missing)

Interpolation	$u(0)$	$u(0.2)$	$u(0.4)$	$u(0.6)$	$u(0.8)$	$u(1)$
$u(3)$	-	-	-	-	-	-
$u(2)$	0.0000	-0.0224	-0.0576	-0.0848	-0.0720	0.0000
$u(1)$	0.0000	0.1728	0.4304	0.7120	0.9216	1.0000
$u(0)$	1.0000	0.9344	0.7264	0.4384	0.1744	0.0000
$u(-1)$	0.0000	-0.0976	-0.1136	-0.0736	-0.0256	0.0000
$u(-2)$	0.0000	0.0128	0.0144	0.0080	0.0016	0.0000

Subdivision	1st fifth	2nd fifth	3rd fifth	4th fifth	5th fifth
$G(2)$	-	-	-	-	-
$G(1)$	-0.0224	-0.0352	-0.0272	0.0128	0.0720
$G(0)$	0.1504	0.2224	0.2544	0.2224	0.1504
$G(-1)$	0.0848	0.0144	-0.0336	-0.0416	-0.0240
$G(-2)$	-0.0128	-0.0016	0.0064	0.0064	0.0016

3. End panel ($u(3)/G(2)$ and $u(2)/G(1)$ are missing)

Interpolation	$u(0)$	$u(0.2)$	$u(0.4)$	$u(0.6)$	$u(0.8)$	$u(1)$
$u(3)$	-	-	-	-	-	-
$u(2)$	-	-	-	-	-	-
$u(1)$	0.0000	0.1280	0.3152	0.5424	0.7776	1.0000
$u(0)$	1.0000	0.9568	0.7840	0.5232	0.2464	0.0000
$u(-1)$	0.0000	-0.0976	-0.1136	-0.0736	-0.0256	0.0000
$u(-2)$	0.0000	0.0128	0.0144	0.0080	0.0016	0.0000

Subdivision	1st fifth	2nd fifth	3rd fifth	4th fifth	5th fifth
$G(2)$	-	-	-	-	-
$G(1)$	-	-	-	-	-
$G(0)$	0.1280	0.1872	0.2272	0.2352	0.2224
$G(-1)$	0.0848	0.0144	-0.0336	-0.0416	-0.0240
$G(-2)$	-0.0128	-0.0016	0.0064	0.0064	0.0016

Note: For the other end, apply the same coeficients as indicated in 2 and 3 with transformation $w(i) = u(1-i)$

Chapter 6

An Interpolation of Abridged Mortality Rates

This note presents a method to construct single-year mortality rates out of the 5-year abridged mortality rates.

1. Problem

Let $q(x;n)$ denote the probability that a life-aged-x will die within n years. In particular, $q(x;1)$ (the mortality rate at the age x) is denoted by $q(x)$.

Let $l(x)$ be the number of lives which are expected to survive to the age x under the assumed mortality. By the definitions of $q(x;n)$ and $l(x)$, we obtain the following relation:

$$l(x+n) = l(x)(1 - q(x;n)).$$

In terms of this notation, the problem is stated as follows: "Suppose 5-year mortality rates at quinquennial ages, $q(0;5), q(5;5), q(10;5), \ldots$, are given, then find the single year mortality rates, $q(0), q(1), q(2), \ldots$".

2. Interpolation Formulae

Let $5p$ denote the ultimate age (e.g. $p = 20, 5p = 100$). In the following, we develop the formulae for four age intervals.

2.1 Ages from 15 to 5p-11

Put

$$G(x;n) = -\ln(1 - q(x;n)) \tag{1a}$$

and

$$G(x) = -\ln(1 - q(x)) \tag{1b}$$

where $\ln x$ is the natural logarithm of x.

Noting that (1b) gives a one-to-one correspondence between $q(x)$ and $G(x)$, we apply the polynomial interpolation to $G(x;n)$ to find $G(x)$. When $G(x)$ is obtained, $q(x)$ is given by solving (1b).

Put

$$\begin{aligned}
u_{-2} &= 0;\\
u_{-1} &= G(5k-10,5);\\
u_0 &= G(5k-10,5) + G(5k-5,5);\\
u_1 &= G(5k-10,5) + G(5k-5,5) + G(5k,5);\\
u_2 &= G(5k-10,5) + G(5k-5,5) + G(5k,5) + G(5k+5,5);\\
u_3 &= G(5k-10,5) + G(5k-5,5) + G(5k,5) + G(5k+5,5) + G(5k+10,5).
\end{aligned}$$

The polynomial interpolating these data are given by $u_x = (1+\Delta)^{x+2}u_{-2}$. Thus,

$$\begin{aligned}
u_x = u_{-2} &+ (x+2)\Delta u_{-2} + \frac{1}{2}(x+2)(x+1)\Delta^2 u_{-2} + \frac{1}{6}(x+2)(x+1)x\Delta^3 u_{-2}\\
&+ \frac{1}{24}(x+2)(x+1)x(x-1)\Delta^4 u_{-2} + \frac{1}{120}(x+2)(x+1)x(x-1)(x-2)\Delta^5 u_{-2}.
\end{aligned}$$

By substituting the finite difference terms in the above formula, we can write it as a linear combination of the data points. The interpolated data for $G(x)$ is given by $G(5k+i) = u_{(i+1)/5} - u_{i/5}$ (for $i = 0, 1, 2, 3, 4$).

The results are shown as follows:

$$\begin{aligned}
G(5k) = \quad & 1/15625 \quad (-126G(5k-10;5) + 1029G(5k-5;5) + 2794G(5k;5)\\
& \qquad -671G(5k+5;5) + 99G(5k+10;5)),\\
G(5k+1) = \quad & 1/15625 \quad (-56G(5k-10;5) + 349G(5k-5;5) + 3289G(5k;5)\\
& \qquad -526G(5k+5;5) + 69G(5k+10;5)),\\
G(5k+2) = \quad & 1/15625 \quad (14G(5k-10;5) - 181G(5k-5;5) + 3459G(5k;5)\\
& \qquad -181G(5k+5;5) + 14G(5k+10;5)),\\
G(5k+3) = \quad & 1/15625 \quad (69G(5k-10;5) - 526G(5k-5;5) + 3289G(5k;5)\\
& \qquad + 349G(5k+5;5) - 56G(5k+10;5)),\\
G(5k+4) = \quad & 1/15625 \quad (99G(5k-10;5) - 671G(5k-5;5) + 2794G(5k;5)\\
& \qquad +1029G(5k+5;5) - 126G(5k+10;5)),
\end{aligned}$$

(for $k = 3, 4, ..., p-3$).

Finally, we obtain the values of $q(x)$ by using the inverse formula of (1b):

$$q(x) = 1 - \exp(-G(x)). \tag{1c}$$

Remark. The assumption underlying the above method is explained as follows. By using the force of mortality

$$m(x) = -\frac{l'(x)}{l(x)}, \tag{2}$$

(1a) and (1b) are rewritten

$$\int_x^{x+n} m(t)dt = G(x;n), \tag{3a}$$

and

$$\int_x^{x+1} m(t)dt = G(x). \tag{3b}$$

Therefore, the above interpolation involves the assumption that the force of mortality can be approximated by a quartic function: $m(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$ from $5k - 10$ to $5k + 15$.

2.2 Ages from 5 to 14

The above method needs to be modified for this age group because required data are not available in the first data panel. Assuming that $q(4)$, hence $G(4)$, is known, we apply polynomial interpolation for the following data:

$$\begin{aligned}
u_4 &= 0;\\
u_5 &= G(4);\\
u_{10} &= G(4) + G(5,5);\\
u_{15} &= G(4) + G(5,5) + G(10,5);\\
u_{20} &= G(4) + G(5,5) + G(10,5) + G(15,5);\\
u_{25} &= G(4) + G(5,5) + G(10,5) + G(15,5) + G(20,5).
\end{aligned}$$

For this, we take the approach described in the remark in 2.1. Assume that the force of mortality from 4 to 25 years of age can be approximated by a quartic function: $m(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$. By substitution, the left-hand-side of equation (3a) will be a polynomial of x, n, a, b, c, d, e. By putting $(x, n) = (4, 1), (5, 5), (10, 5), (15, 5), (20, 5)$ in (3a), we shall obtain five equations of five unknown variables a, b, c, d, e. By substituting the solution of this system of equations into equation (3b), we obtain $G(x)$ for $x = 5, 6, 7, ..., 14$.

The results are as follows:

$$
\begin{array}{lll}
G(5) = & 1/577500 & (249375G(4) + 89523G(5;5) - 33369G(10;5) \\
& & +11319G(15;5) - 1848G(20;5)), \\
G(6) = & 1/577500 & (43125G(4) + 131829G(5;5) - 33567G(10;5) \\
& & +10197G(15;5) - 1584G(20;5)), \\
G(7) = & 1/577500 & (-69375G(4) + 139449G(5;5) - 12087G(10;5) \\
& & +2277G(15;5) - 264G(20;5)), \\
G(8) = & 1/577500 & (-113125G(4) + 123419G(5;5) + 21163G(10;5) \\
& & -7733G(15;5) + 1276G(20;5)), \\
G(9) = & 1/577500 & (-110000G(4) + 93280G(5;5) + 57860G(10;5) \\
& & -16060G(15;5) + 2420G(20;5)), \\
G(10) = & 1/577500 & (-78750G(4) + 57078G(5;5) + 91266G(10;5) \\
& & -19866G(15;5) + 2722G(20;5)), \\
G(11) = & 1/577500 & (-35000G(4) + 21364G(5;5) + 116228G(10;5) \\
& & -17248G(15;5) + 2156G(20;5)), \\
G(12) = & 1/577500 & (8750G(4) - 8806G(5;5) + 129178G(10;5) \\
& & -7238G(15;5) + 616G(20;5)), \\
G(13) = & 1/577500 & (43125G(4) - 29871G(5;5) + 128133G(10;5) \\
& & +10197G(15;5) - 1584G(20;5)), \\
G(14) = & 1/577500 & (61875G(4) - 39765G(5;5) + 112695G(10;5) \\
& & +34155G(15;5) - 3960G(20;5)).
\end{array}
$$

We obtain the values of $q(x)$ by (1c).

Remark. The minimum age of the workers covered by social security schemes is usually higher than 15 years, therefore for the purposes of actuarial valuation mortality rates of this age class are not necessary.

2.3 Ages from 0 to 4

A special consideration is needed in view of the relatively high mortality rates of infants. For this reason, single year mortality rates are usually given for ages under 5 years. In case no single year mortality rates under age 5 are available, the UN model life tables include figures of $q(0), q(1), .., q(4)$ for various mortality patterns.

As in the above case, this age group is outside the coverage of social security pension schemes.

2.4 Ages over 5p-10

For higher ages, we assume that the mortality force follows the Gompertz-Makeham's law: $m(x) = A + BC^x$.

Substituting the above expression of $m(x)$ for (3a) and (3b), we get

$$G(x;5) = 5A + BC^x(C^5 - 1)/\ln C, \tag{4a}$$

and

$$G(x) = A + BC^x(C - 1)/\ln C. \tag{4b}$$

Putting $x = 5p - 15, 5p - 10, 5p - 5$ in (4a) will yield three equations of three unknown variables A, B and C. By using the solution of these equations, we get $G(x)$ from (4b). The result is as follows:

$$\begin{aligned} G(x) = \{G(5p-5;5) - G(5p-10;5)\} \cdot (C-1) \cdot C^{x-5p-10}/(C^5-1)^2 \\ + [G(5p-10;5) - \{G(5p-5;5) - G(5p-10;5)\}/(C^5-1)]/5, \quad (5) \end{aligned}$$

where

$$C = \sqrt[5]{\frac{G(5p-5;5) - G(5p-10;5)}{G(5p-10;5) - G(5p-15;5)}}.$$

At and above the age $5p - 1$ (=99, if $5p = 100$), we put $q(x) = 1.0$.

3. Application

We shall apply this method to the ultimate life tables developed by the UN.

(i) The ultimate life tables (abridged by 5 years)

Table 1 shows the ultimate life tables calculated by the UN. They are constructed by selecting the lowest mortality rates of the world for each age class. The life expectancy at birth is 82.5 for males, and 87.5 for females. These tables give the 5-year mortality rates of each quinquennial age.

(ii) The interpolated ultimate life tables

Table 2 shows the ultimate life tables for single ages.

- Under 5 years old, only the mortality rates at the age 0 and that from 1 to 5 are given in the original table. For the ages 1 to 4, the mortality rates are obtained by simple exponential interpolation.
- For ages 5 to 14, the formulae developed in 2.2 are applied.
- For ages 15 to 89, the formulae developed in 2.1 are applied.
- For ages over 90, the formula developed in 2.4 is applied.
- It is observed that the 5-year accumulated mortality rates of every quinquennial age are, in fact, equal to the 5-year mortality rates in the original tables.
- The life expectancies at birth calculated from these unabridged tables are 82.41 for males and 87.26 for females.

References

Ministry of Health and Welfare of Japan. (1960). *Life Tables.*

United Nations. (1982). *Unabridged Model Life Tables Corresponding to the New United Nations Model Life Tables for Developing Countries.*

United Nations. (1988, 1990). *World Population Prospects.*

Table 1. Construction of the abridged ultimate mortality rates

	Low mortality countries		Model ultimate life table	
Age	Males	Females	Males	Females
0	0.00584 b	0.00499 b	0.00500	0.00400
1-5	0.00113 a	0.00088 f	0.00500	0.00030
5-10	0.00118 f	0.00062 f	0.00040	0.00020
10-15	0.00095 f	0.00024 a	0.00025	0.00015
15-20	0.00301 c	0.00117 b	0.00055	0.00035
20-25	0.00388 c	0.00138 d	0.00090	0.00050
25-30	0.00380 c	0.00165 d	0.00130	0.00070
30-35	0.00442 c	0.00246 d	0.00160	0.00100
35-40	0.00652 c	0.00393 b	0.00220	0.00150
40-45	0.00952 c	0.00598 a	0.00430	0.00300
45-50	0.01747 a	0.00918 b	0.00900	0.00460
50-55	0.03081 b	0.01203 a	0.01300	0.00600
55-60	0.04493 b	0.02064 b	0.02350	0.01040
60-65	0.06525 b	0.03267 b	0.03480	0.01640
65-70	0.10335 b	0.05469 b	0.05500	0.02800
70-75	0.16918 b	0.09493 b	0.09000	0.05000
75-80	0.26794 a	0.17906 b	0.14500	0.09000
80-85	0.35825 a	0.28990 a	0.23700	0.15000
85-90	0.60335 c	0.45508 e	0.37100	0.25000
90-95	0.74593 e	0.59852 e	0.57000	0.43000
95-100	0.85474 c	0.82426 c	0.77500	0.71000

Source: World Population Prospects 1988, United Nations.
Notes: a-Iceland, 1983; b-Japan, 1985-86; c-Netherlands, 1983-84; d-Norway, 1982-83; e-Switzerland, 1981-82; f-Sweden, 1983

Table 2. The ultimate mortality rates by single year

	Males				Males(cont'd)				Females				Females(cont'd)		
Age	$Q(x)$	$l(x)$	$e^{\circ}(x)$	Age	$Q(x)$	$l(x)$	$e^{\circ}(x)$	Age	$Q(x)$	$l(x)$	$e^{\circ}(x)$	Age	$Q(x)$	$l(x)$	$e^{\circ}(x)$
0	0.00500	100,000	82.44	51	0.00229	97,217	32.86	0	0.00400	100,000	87.35	51	0.00108	98,281	37.34
1	0.00014	99,500	81.86	52	0.00255	96,994	31.94	1	0.00008	99,600	86.69	52	0.00117	98,175	36.38
2	0.00014	99,486	80.87	53	0.00287	96,747	31.02	2	0.00008	99,593	85.70	53	0.00130	98,060	35.42
3	0.00014	99,473	79.88	54	0.00325	96,469	30.10	3	0.00008	99,585	84.71	54	0.00145	97,933	34.47
4	0.00014	99,459	78.89	55	0.00387	96,156	29.20	4	0.00008	99,578	83.71	55	0.00170	97,790	33.52
5	0.00011	99,445	77.90	56	0.00431	95,783	28.31	5	0.00006	99,570	82.72	56	0.00189	97,624	32.57
6	0.00009	99,434	76.91	57	0.00475	95,370	27.43	6	0.00005	99,564	81.72	57	0.00208	97,440	31.63
7	0.00008	99,424	75.92	58	0.00518	94,917	26.56	7	0.00004	99,559	80.73	58	0.00228	97,238	30.70
8	0.00006	99,417	74.92	59	0.00561	94,425	25.70	8	0.00003	99,556	79.73	59	0.00250	97,016	29.77
9	0.00005	99,411	73.93	60	0.00594	93,896	24.84	9	0.00003	99,553	78.73	60	0.00270	96,773	28.84
10	0.00005	99,405	72.93	61	0.00643	93,338	23.99	10	0.00002	99,550	77.74	61	0.00296	96,512	27.92
11	0.00004	99,401	71.93	62	0.00698	92,737	23.14	11	0.00002	99,548	76.74	62	0.00326	96,227	27.00
12	0.00005	99,397	70.94	63	0.00761	92,090	22.30	12	0.00003	99,545	75.74	63	0.00360	95,913	26.08
13	0.00005	99,392	69.94	64	0.00833	91,389	21.46	13	0.00003	99,543	74.74	64	0.00399	95,568	25.18
14	0.00006	99,387	68.94	65	0.00916	90,628	20.64	14	0.00004	99,539	73.74	65	0.00447	95,186	24.28
15	0.00008	99,381	67.95	66	0.01008	89,798	19.83	15	0.00005	99,535	72.75	66	0.00499	94,761	23.38
16	0.00010	99,373	66.95	67	0.01112	88,893	19.02	16	0.00006	99,530	71.75	67	0.00558	94,288	22.50
17	0.00011	99,363	65.96	68	0.01229	87,905	18.23	17	0.00007	99,524	70.76	68	0.00626	93,762	21.62
18	0.00012	99,352	64.97	69	0.01360	86,825	17.45	18	0.00008	99,517	69.76	69	0.00703	93,176	20.75
19	0.00014	99,340	63.98	70	0.01517	85,644	16.68	19	0.00008	99,509	68.77	70	0.00786	92,521	19.90
20	0.00015	99,326	62.98	71	0.01677	84,344	15.93	20	0.00009	99,500	67.77	71	0.00888	91,794	19.05
21	0.00016	99,311	61.99	72	0.01851	82,930	15.20	21	0.00009	99,492	66.78	72	0.01004	90,979	18.22
22	0.00018	99,295	61.00	73	0.02042	81,395	14.47	22	0.00010	99,482	65.78	73	0.01137	90,066	17.40
23	0.00020	99,277	60.01	74	0.02254	79,733	13.77	23	0.00011	99,473	64.79	74	0.01287	89,041	16.59
24	0.00021	99,258	59.03	75	0.02478	77,936	13.07	24	0.00011	99,462	63.80	75	0.01475	87,895	15.80
25	0.00023	99,237	58.04	76	0.02744	76,005	12.39	25	0.00012	99,451	62.80	76	0.01656	86,599	15.03
26	0.00025	99,214	57.05	77	0.03045	73,920	11.73	26	0.00013	99,439	61.81	77	0.01851	85,165	14.27
27	0.00026	99,189	56.07	78	0.03384	71,669	11.08	27	0.00014	99,426	60.82	78	0.02063	83,589	13.53
28	0.00027	99,163	55.08	79	0.03767	69,243	10.45	28	0.00015	99,412	59.83	79	0.02296	81,864	12.81
29	0.00029	99,136	54.10	80	0.04244	66,635	9.84	29	0.00016	99,397	58.84	80	0.02573	79,985	12.10
30	0.00030	99,108	53.11	81	0.04708	63,807	9.25	30	0.00018	99,381	57.85	81	0.02852	77,927	11.40
31	0.00031	99,078	52.13	82	0.05213	60,804	8.69	31	0.00019	99,364	56.86	82	0.03159	75,704	10.72
32	0.00032	99,048	51.14	83	0.05768	57,634	8.14	32	0.00020	99,345	55.87	83	0.03504	73,313	10.06
33	0.00033	99,016	50.16	84	0.06384	54,310	7.60	33	0.00021	99,325	54.88	84	0.03897	70,744	9.40
34	0.00035	98,983	49.17	85	0.07008	50,843	7.09	34	0.00023	99,304	53.89	85	0.04409	67,987	8.77
35	0.00036	98,949	48.19	86	0.07803	47,279	6.58	35	0.00023	99,282	52.90	86	0.04914	64,990	8.15
36	0.00039	98,913	47.21	87	0.08717	43,590	6.10	36	0.00026	99,259	51.91	87	0.05492	61,796	7.54
37	0.00043	98,875	46.23	88	0.09759	39,790	5.63	37	0.00029	99,234	50.93	88	0.06164	58,402	6.95
38	0.00048	98,832	45.25	89	0.10937	35,907	5.19	38	0.00034	99,205	49.94	89	0.06956	54,802	6.37
39	0.00054	98,785	44.27	90	0.12346	31,980	4.76	39	0.00039	99,171	48.96	90	0.07910	50,990	5.81
40	0.00060	98,731	43.29	91	0.13784	28,032	4.37	40	0.00047	99,133	47.98	91	0.09038	46,957	5.27
41	0.00070	98,672	42.32	92	0.15357	24,168	3.98	41	0.00053	99,086	47.00	92	0.10371	42,712	4.74
42	0.00084	98,603	41.35	93	0.17072	20,456	3.61	42	0.00060	99,033	46.02	93	0.11941	38,283	4.24
43	0.00099	98,520	40.38	94	0.18939	16,964	3.26	43	0.00067	98,974	45.05	94	0.13785	33,711	3.74
44	0.00118	98,422	39.42	95	0.20966	13,751	2.90	44	0.00073	98,908	44.08	95	0.15942	29,064	3.26
45	0.00147	98,307	38.47	96	0.23161	10,868	2.54	45	0.00082	98,835	43.11	96	0.18455	24,431	2.78
46	0.00165	98,162	37.52	97	0.25532	8,351	2.15	46	0.00088	98,755	42.15	97	0.21365	19,922	2.30
47	0.00182	98,000	36.58	98	0.28083	6,219	1.72	47	0.00093	98,668	41.18	98	0.24713	15,666	1.79
48	0.00197	97,821	35.65	99	0.30818	4,472	1.19	48	0.00097	98,577	40.22	99	0.28536	11,794	1.21
49	0.00211	97,628	34.72	100	1.00000	3,094	0.50	49	0.00102	98,481	39.26	100	1.00000	8,429	0.50
50	0.00210	97,422	33.79					50	0.00101	98,381	38.30				

Chapter 7

The Concept of Stable Population

A limit population under constant demographic assumptions

Introduction

In this note we present a result concerning the long-term behaviour of the population dynamics. It asserts that for a closed population (i.e., no migration is assumed) if birth rates and mortality rates remain constant then, in the long run, the growth rate of the population will converge to a certain constant rate, and its age distribution will converge to a certain distribution. The ultimate growth rate and age distribution are characterized by the given birth and mortality rates and do not depend on the initial population. This limit population is called the stable population.

The above statement is mathematically formulated as follows. For simplicity, we consider a unisex population and denote it by $L(x,t)$ (x: age, t: time). The stable population, denoted by $P_S(x,t)$, is expressed in the following form:

$$P_S(x,t) = N(t) \cdot C(x),$$

where $N(t)$ is the size of the total population growing at a certain rate r, i.e. $N(t) = Ae^{rt}$. $C(x)$ is the age distribution independent of time and is written: $C(x) = Be^{-rx}p(x)$.

The stable population satisfies the relation:

$$\lim_{t\to\infty} \frac{L(x,t)}{N(t)} = \frac{P_S(x,t)}{N(t)} = C(x).$$

In the above,

$p(x)$	:	Probability that a new life will survive until age x (in the standard actuarial notation, ${}_xp_0$) (assumed constant over time)
$b(x)$	:	Birth rate of age x (assumed constant over time)
r	:	Intrinsic growth rate, which is the (unique) real root of the following equation: $\int_0^\infty e^{-rx}b(x)p(x)dx = 1$.
A	:	Constant depending on the initial population, birth rates, and mortality rates.
B	:	Constant depending on the birth rates and mortality rates (but not on the initial population). By the normalization condition, $\int_0^\infty C(x)dx = 1$. B is calculated as: $B = \left(\int_0^\infty e^{-rx}p(x)dx\right)^{-1}$.

We shall show this by continuous and discrete approaches.

1. Continuous approach

1.1. Let[1]

$u(x,t)$	:	Population of age x at time t.
$b(x,t)$	:	Birth rate of age x at time t.
$d(x,t)$	:	Mortality force of age x at time t.

Then, the following equation holds:

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = -d(x,t)u.$$

Proof. By the definition of birth rates, the number of newborn, $u(0,t)$, is given by

$$u(0,t) = \int_0^\infty b(x,t)u(x,t)dx.$$

By the definition of mortality force, the change in the population in a time step of h is given by

$$u(x+h,t+h) - u(x,t) = -d(x,t)\cdot h\cdot u(x,t).$$

On the other hand, by Taylor's theorem, we have

$$u(x+h,t+h) - u(x,t) = \frac{\partial u}{\partial x}(x,t)h + \frac{\partial u}{\partial t}(x,t)h + O(h^2).$$

Combining the above results and dividing each side by h, we have,

$$\frac{u(x+h,t+h) - u(x,t)}{h} = -d(x,t)u(x,t) = \frac{\partial u}{\partial x}(x,t) + \frac{\partial u}{\partial t}(x,t) + \frac{O(h^2)}{h}.$$

[1] We assume that these functions are differentiable for any required times

Taking the limit $h \to 0$ will yield the required equation. (Q.E.D.)

1.2. Consider the initial value problem:

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = -d(x,t)u, \tag{1}$$

with $u(0,t) = B(t)$, the number of newborns for each year, and, $u(x,0) = u_0(x)$, population at the initial time.

Solution. To solve this equation, introduce a characteristic coordinate s (cohort line) such that

$$\frac{dt}{ds} = 1, \qquad \frac{dx}{ds} = 1.$$

Then,

$$\frac{du}{ds} = \frac{\partial u}{\partial t}\frac{dt}{ds} + \frac{\partial u}{\partial x}\frac{dx}{ds} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}.$$

(i) For the population existing at the initial time.

Let x_0 denote the age at $t = 0$. Put $t = s$ and $x = s + x_0, (x_0 \geqq 0)$, then,

$$\frac{du}{ds} = -d(s + x_0, s)u, \qquad \text{with } u\big|_{s=0} = u_0(x_0).$$

The integral of this equation is given by

$$u(x_0 + s, s) = u_0(x_0) \exp\left(-\int_0^s d(\sigma + x_0, \sigma)d\sigma\right).$$

Eliminating s and x_0, (by $s = t, x_0 = x - t$), we obtain

$$u(x,t) = u_0(x-t) \exp\left(-\int_0^t d(\sigma + x - t, \sigma)d\sigma\right) \qquad \text{for } x \geqq t.$$

(ii) For the population to be born in the future.

Let t_0 denote the year of birth. Put $t = s + t_0$ and $x = s$, $(t_0 > 0)$, then,

$$\frac{du}{ds} = -d(s, s + t_0)u, \qquad \text{with } u\big|_{s=0} = B(t_0).$$

Thus, we have

$$u(s, t_0 + s) = B(t_0) \exp\left(-\int_0^s d(\sigma, \sigma + t_0)d\sigma\right).$$

Eliminating s and t_0 (by $x = s, t_0 = t - x$), we have

$$u(x,t) = B(t-x) \exp\left(-\int_0^x d(\sigma, \sigma + t - x)d\sigma\right), \qquad \text{for } x < t.$$

1.3. The initial value problem (1) in 1.2 is attributed to the following single integral equation for $B(t)$:

$$B(t) = f(t) + \int_0^t k(x,t)B(t-x)dx. \tag{2}$$

Proof. By definition,

$$B(t) = \int_0^\infty b(x,t)u(x,t)dx.$$

By substituting the solution to (1) into this, we have

$$\begin{aligned} B(t) = \int_0^t b(x,t)B(t-x)\ \exp\left(-\int_0^x d(\sigma,\sigma+t-x)d\sigma\right) dx \\ + \int_t^\infty b(x,t)u_0(x-t)\ \exp\left(-\int_0^t d(\sigma+x-t,\sigma)d\sigma\right) dx. \end{aligned}$$

Putting

$$f(t) = \int_t^\infty b(x,t)u_0(x-t)\ \exp\left(-\int_0^t d(\sigma+x-t,\sigma)d\sigma\right) dx,$$

and

$$k(x,t) = b(x,t)\ \exp\left(-\int_0^x d(\sigma,\sigma+t-x)d\sigma\right)$$

will yield the above equation (2). (Q.E.D.)

Remark. Let M be any upper bound of reproductive ages (i.e. $b(x,t)=0$ for $x \geqq M$ and all t), then it follows that $f(t)=0$ for $t>M$.

1.4. For further investigation into the problem (2), we describe some mathematical tools.

(i) The Laplace transform of a function $B(t)$ is defined as

$$\widehat{B}(s) = \int_0^\infty e^{-st}B(t)dt.$$

Here, $s=\sigma+i\tau$ is a complex variable. If the improper integral defining $\widehat{B}(s)$ converges for a value $s=s_c$, then it converges for all s with Re $(s) >$ Re (s_c).

(ii) The inverse formula for the Laplace transform is given in the following form:

$$B(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\widehat{B}(s)ds,$$

where the contour of integration is chosen to lie within the domain of convergence of $\widehat{B}(s)$.

A formal proof of the above formula will go as follows:

$$\begin{aligned}
\int_{c-i\infty}^{c+i\infty} e^{st}\widehat{B}(s)ds &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\left(\int_0^\infty e^{-s\tau}B(\tau)d\tau\right)ds \\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\int_{-\infty}^{\infty} e^{s(t-\tau)}B(\tau)d\tau ds \\
&= \frac{e^{ct}}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} e^{i\xi(t-\tau)}d\xi\right)e^{-c\tau}B(\tau)d\tau \qquad (s=c+i\xi) \\
&= e^{ct}\int_{-\infty}^{\infty}\delta(t-\tau)e^{-c\tau}B(\tau)d\tau = B(t).
\end{aligned}$$

1.5. Now assume that mortality rates (hence, mortality forces) and birth rates depend on age only, and not on time, i.e. functions d and b depend only on x, not on t. Then function k depends only on x. Thus, we have

$$B(t) = f(t) + \int_0^t k(x)B(t-x)dx \qquad \text{(renewal equation)}, \tag{3}$$

where

$$k(x) = b(x)\exp\left(-\int_0^x d(\sigma)d\sigma\right) = b(x)p(x).$$

To solve equation (3), we first show the following relation:

$$\widehat{B}(s) = \frac{\widehat{f}(s)}{1-\widehat{k}(s)}.$$

Proof. Write

$$\widehat{B}(s) = \int_0^\infty e^{-st}B(t)dt = \int_0^\infty e^{-st}f(t)dt + \int_0^\infty\left(\int_0^t k(x)B(t-x)dx\right)e^{-st}dt.$$

By changing the order of integration, and putting $y = t-x$, we have

$$\begin{aligned}
\widehat{B}(s) &= \widehat{f}(s) + \int_0^\infty k(x)e^{-sx}dx \cdot \int_0^\infty B(y)e^{-sy}dy \\
\therefore \widehat{B}(s) &= \widehat{f}(s) + \widehat{k}(s)\cdot\widehat{B}(s).
\end{aligned}$$

This completes the proof. (Q.E.D.)

Remark. Equation (3) is written by $B = f + k * B$, where $*$ denotes the convolution. As this can be rewritten as $(1-k)*B = f$, a formal solution is $B = \sum_{n=0}^{\infty}(k^{*n}) * f = f + k*f + k*k*f + \cdots$. We can approximate this solution by the following iteration: $B_0 = f$ and $B_{j+1} = f + k * B_j$ (for $j \geqq 0$).

1.6. The equation $1-\widehat{k}(s) = 0$ is called the characteristic equation. This equation has a unique simple real root, denoted by r, and all other complex roots have real parts less than r.

Proof. First, we note that

$$\widehat{k}'(s) = \frac{d\widehat{k}(s)}{ds} = -\int_0^\infty xk(x)e^{-sx}dx < 0.$$

Since $\lim_{s\to-\infty}\widehat{k}(s) = \infty$ and $\lim_{s\to\infty}\widehat{k}(s) = 0$, the equation $1-\widehat{k}(s) = 0$ has a unique simple real root r.

Second, suppose that $s = p + iq$ ($q \neq 0$) is a complex root of the characteristic equation, then

$$\widehat{k}(p+iq) = \int_0^\infty k(x)e^{-(p+iq)x}dx = \int_0^\infty k(x)e^{-px}\cos(qx)dx - i\int_0^\infty k(x)e^{-px}\sin(qx)dx = 1.$$

Hence,

$$\int_0^\infty k(x)e^{-rx}dx = \int_0^\infty k(x)e^{-px}\cos(qx)dx = 1.$$

Since $q \neq 0$, assuming $r \leqq p$ will lead to a contradiction with the above relation. Therefore, $r > p$. (Q.E.D.)

Remarks.

(i) Since $k(x) = b(x)p(x)$. The characteristic equation is given by:

$$1-\widehat{k}(s) = 1-\int_0^\infty e^{-sx}b(x)p(x)dx = 0.$$

Later we shall see that the above root of the characteristic equation $s = r$ is the intrinsic growth rate (see Introduction).

(ii) If the characteristic equation has a complex root $s = p+iq$, then its complex conjugate $\overline{s} = p-iq$ is also a root. (Thus, in the complex plane, the roots of the characteristic equation lie symmetrically with respect to the real axis).

1.7. Assume that the characteristic equation has only simple roots. We denote them by $\{s_m\}$, where $s_0 = r$ is real, and the other s_m (for $m \geqq 1$) are complex and Re $(s_m) < s_0$. Then,

$$B(t) = \sum_{m=0}^{\infty} A_m e^{s_m t}. \tag{4}$$

Proof. By the inversion formula of the Laplace transform, we have for a fixed $\gamma > s_0$

$$B(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\widehat{B}(s)ds = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{\widehat{f}(s)}{1-\widehat{k}(s)}ds.$$

By changing the contour of integration, the integral breaks up into a sum of integrals (★),

$$B(t) = \sum_{m=0}^{\infty}\frac{1}{2\pi i}\int_{c_m} e^{st}\frac{\widehat{f}(s)}{1-\widehat{k}(s)}ds = \sum_{m=0}^{\infty}\mathrm{Res}_{s=s_m} e^{st}\frac{\widehat{f}(s)}{1-\widehat{k}(s)}ds,$$

where c_m is a circle with a (sufficiently small) radius $\rho_m > 0$ around s_m, i.e. $c_m = \{s_m + \rho_m e^{i\theta}; 0 \leqq \theta < 2\pi\}$.

By assumption, the integrand has only poles of order 1, thus

$$\begin{aligned}\mathrm{Res}_{s=s_m} e^{st}\frac{\widehat{f}(s)}{1-\widehat{k}(s)}ds &= \lim_{s\to s_m}(s-s_m)e^{st}\frac{\widehat{f}(s)}{1-\widehat{k}(s)} \\ &= -\lim_{s\to s_m} e^{st}\widehat{f}(s)\left(\frac{\widehat{k}(s)-\widehat{k}(s_m)}{s-s_m}\right)^{-1} = -e^{s_m t}\frac{\widehat{f}(s_m)}{\widehat{k}'(s_m)}.\end{aligned}$$

Hence, putting

$$A_m = -\frac{\widehat{f}(s_m)}{\widehat{k}'(s_m)}$$

will give the required formula. (Q.E.D.)

Remarks.

(i) In view of remark of 1.6, we rearrange the order of the roots of the characteristic equation $\{s_m\}$, such that $s_0 = r$ is real and s_{2m-1} and $s_{2m}(m \geqq 1)$ are mutually conjugate (we denote these complex roots by $p_m \pm iq_m$ with $|p_m| < r$). Then, the coefficients A_{2m-1} and A_{2m} are also mutually conjugate (denoted by $\alpha_m \exp(\pm i\beta_m), \alpha_m = |A_{2m-1}| = |A_{2m}|$).

(ii) Therefore, by eliminating complex numbers, we have the following "real" expression of $B(t)$:

$$B(t) = A_0 e^{rt} + \sum_{m=1}^{\infty} 2\alpha_m e^{p_m t}\cos(q_m t + \beta_m).$$

(iii) The above argument in ($\bigstar$) is not rigorous. However, the proposition in 1.8 (our main result) will still hold. In fact, the following facts are known[2]:

a) For any $\sigma \in \mathbb{R}$, there are finite number of roots of the characteristic equation in $\{s \in \mathbb{C};\ \mathrm{Re}\ (s) > \sigma\}$

b) In addition to (i), rearrange the order of the roots of the characteristic equation $\{s_m\}$ by the decending order of their real parts, and choose $\delta \in \mathbb{R}$ such that $r = s_0 > \mathrm{Re}\ (s_1) = \mathrm{Re}\ (s_2) \geqq \cdots \geqq \mathrm{Re}\ (s_{2\nu-1}) = \mathrm{Re}\ (s_{2\nu}) > \delta > \mathrm{Re}\ (s_{2\nu+1})$. Further, assume that the order of s_m is $h_m \geqq 1$. Then we have

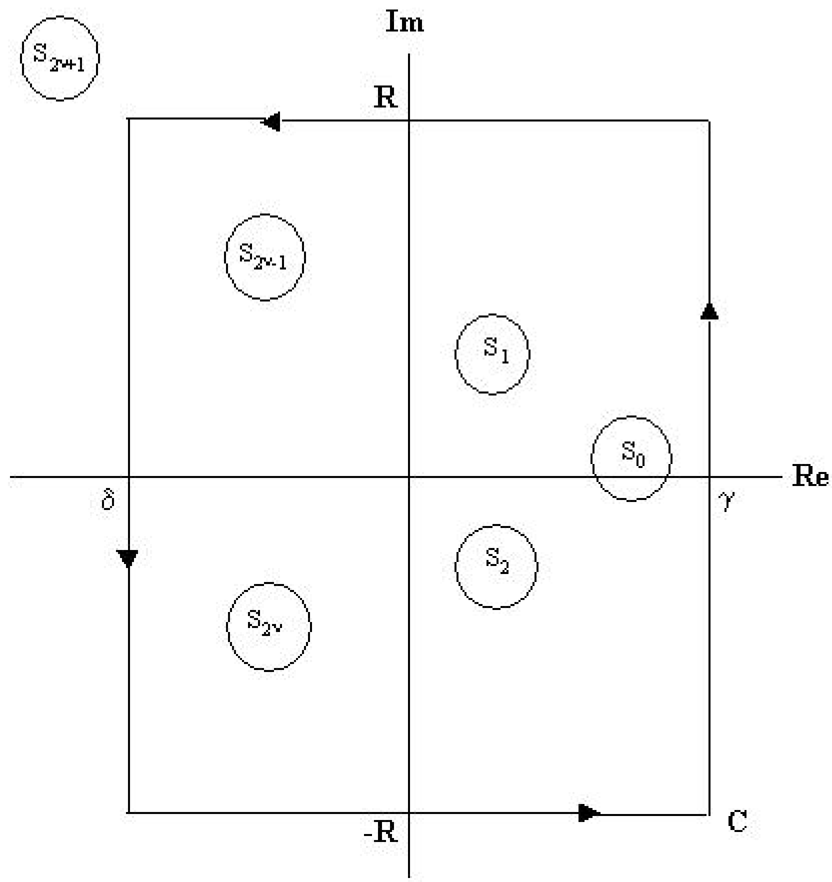
$$B(t) = f(t) + \sum_{m=0}^{2\nu} e^{s_m t}\left(\sum_{j=1}^{h_m} A_j^{(m)} t^{j-1}\right) + O(e^{\delta t}),$$

where

$$A_j^{(m)} = \frac{1}{(j-1)!}\int_{c_m}(z-s_m)^{j-1}\left(\widehat{B}(z)-\widehat{f}(z)\right)dz.$$

[Outline of the Proof.] First, we note that we can find such δ and ν in view of a). Apply Cauchy's formula for the function $e^{zt}\left(\widehat{B}(z)-\widehat{f}(z)\right)$ on the rectangular contour linking

[2]For the detail, see Inaba in the references.



$\gamma - iR$, $\gamma + iR$, $\delta + iR$, $\delta - iR$ (choose $\gamma > r$ and $R > \max\{|\mathrm{Im}(s_m)|;\ 0 \leqq m \leqq 2\nu\}$). As $R \to \infty$, we can show that the contribution from vertical lines will tend to zero (Riemann-Lebesgues) and that the contribution from the vertical line passing δ is dominated by $O(\delta)$. Then apply the inversion formula for the integral on the vertical line passing γ.

1.8. For each x,

$$\lim_{t\to\infty} e^{-rt} u(x,t) = A_0 e^{-rx} p(x).$$

Therefore, further to the above notation, putting

$$A = A_0 \int_0^\infty e^{-rx} p(x)dx \quad ; \quad B = \left(\int_0^\infty e^{-rx} p(x)dx\right)^{-1}$$

will lead to the population $P_S(x,t) = N(t) \cdot C(x)$, where $N(t) = Ae^{rt}$ and $C(x) = Be^{-rx}p(x)$, such that

$$\lim_{t\to\infty} \frac{u(x,t)}{N(t)} = \frac{P_S(x,t)}{N(t)} = C(x).$$

Proof. Substituting (4) in 1.7 into the solution of (1) in 1.2 (ii), we have

$$u(x,t) = B(t-x)\exp\left(-\int_0^x d(\sigma)d\sigma\right) = \sum_{m=0}^{\infty} A_m e^{s_m(t-x)} p(x).$$

Thus,

$$e^{-rt}u(x,t) = A_0 e^{-rx} p(x) + \sum_{m=1}^{\infty} A_m e^{(s_m - s_0)t} e^{-s_m x} p(x).$$

Since Re $(s_m) < s_0$ (for $m \geqq 1$),

$$\lim_{t\to\infty} |e^{(s_m-s_0)t}| = \lim_{t\to\infty} |e^{(\mathrm{Re}(s_m)-s_0)t}| = 0 \qquad (\text{for } m \geqq 1).$$

Hence,

$$\lim_{t\to\infty} e^{-rt}u(x,t) = A_0 e^{-rx} p(x). \qquad (\text{Q.E.D.})$$

2. Discrete approach

In contrast to the previous part, the existence of stable population is shown here in a discrete case. In addition to the unisex population assumption, we assume that all births and deaths occur at the beginning of each year.

2.1. Let

$l(x,t)$	:	Population of age x at the beginning of year t.
$b(x,t)$	:	Birth rate of age x at the beginning of year t for the newborn at the beginning of year $t+1$.
$p(x,t)$	:	Survival rate that a life aged x at the beginning of year t will survive until the beginning of year $t+1$.

Here, x ranges from 0 to $n-1$ (n is the ultimate age e.g. 100 years). It should be noted that $b(x,t) > 0$ if x is a reproductive age (e.g. from 15 to 49 years) and $b(x,t) = 0$ otherwise.

When the initial population $\{l(x,0)\}$, birth rates $\{b(x,t)\}$ and survival rates $\{p(x,t)\}$ are given, the future population, $\{l(x,t); t \geqq 1\}$, is calculated by the following equations (the cohort composition method):

$$\begin{cases} l(0,t+1) = \displaystyle\sum_{x=0}^{n-1} b(x,t)l(x,t); \\ l(x+1,t+1) = p(x,t)l(x,t) \qquad (\text{for } 0 \leqq x \leqq n-2). \end{cases}$$

Note that the sum in the first equation is effectively taken over reproductive ages.

2.2. Assume that birth rates and survival rates are independent of time, i.e. $b(x,t) = b_x$ and $p(x,t) = p_x$. Then, by using the vector notation, the above equations are written as a linear system:

$$\boldsymbol{l}_{t+1} = A\boldsymbol{l}_t,$$

where, $\boldsymbol{l}_t = (l(t,x))$ is a population vector (column vector notation) at year t, and

$$A = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{n-2} & b_{n-1} \\ p_0 & 0 & 0 & & 0 & 0 \\ 0 & p_1 & 0 & & 0 & 0 \\ 0 & 0 & p_2 & & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \cdots & p_{n-2} & 0 \end{pmatrix}$$

is a transition matrix (also called the Leslie matrix).

Therefore, starting from the population in the initial year, $\boldsymbol{l}_0 \geqq \mathbf{0}$, the population in year t is calculated by : $\boldsymbol{l}_t = A^t \boldsymbol{l}_0$.

2.3. The above-defined transition matrix A has the following properties:

(i) The characteristic polynomial of A, denoted by $\phi_A(\lambda)$, is calculated as

$$\begin{aligned}\phi_A(\lambda) &= \det(\lambda I_n - A) \\ &= \lambda^n - b_0\lambda^{n-1} - p_0 b_1 \lambda^{n-2} - p_0 p_1 b_2 \lambda^{n-3} - \cdots - p_0 p_1 \cdots p_{n-3} p_{n-2} b_{n-1}.\end{aligned}$$

(ii) Further, $\phi_A(\lambda)$ is identical to the minimal polynomial of A.

[Outline of the Proof: Show that the matrix $\lambda I_n - A$ has $\phi_A(\lambda)$ as its elementary divisor].

(iii) $\phi_A(\lambda) = 0$ has a unique, strictly positive, simple real root. Denote this by $r > 0$.

Proof. By dividing both sides of $\phi_A(\lambda) = 0$ by λ^n and rearranging, we have

$$1 = b_0\lambda^{-1} + p_0 b_1 \lambda^{-2} + \cdots + p_0 p_1 \cdots p_{n-3} b_{n-2} \lambda^{-(n-1)} + p_0 p_1 \cdots p_{n-3} p_{n-2} b_{n-1} \lambda^{-n} =: k(\lambda).$$

Since $k(\lambda)$ is a strictly decreasing function for $\lambda \geqq 0$ with $\lim_{s\to 0} k(s) = \infty$ and $\lim_{s\to\infty} k(s) = 0$, we have the required statement. (Q.E.D.)

(iv) The eigenvector associated with r is given

$$\boldsymbol{p} = (1, r^{-1}p_0, r^{-2}p_0p_1, \ldots, r^{-(n-1)}p_0p_1\cdots p_{n-3}p_{n-2})^t,$$

or alternatively,

$$\boldsymbol{p} = ({}_0p_0,\ r^{-1}{}_1p_0,\ r^{-2}{}_2p_0, \ldots, r^{-(n-1)}{}_{n-1}p_0)^t.$$

2.4. Let M denote a matrix or a vector. If all elements of M are non-negative (positive), then M is called non-negative (positive) and denoted by $M \geqq O$ ($M > O$). Concerning non-negative matrices, the following theorem is known.

[Frobenius's theorem] *A non-negative $n \times n$ matrix M has an eigenvalue $\mu \geqq 0$ such that*

(i) *With μ we can associate a non-negative eigenvector $\boldsymbol{m} \geqq \mathbf{0}$.*

(ii) *If* $\rho \in \mathbb{C}$ *is any eigenvalue of* M, *then* $\mu \geqq |\rho|$.

The above eigenvalue μ is called the Frobenius root of the matrix M.-

Since the transition matrix A is non-negative, we can apply the Frobenius's theorem to A. From the results in 2.3, it follows that $r > 0$ is the Frobenius root of A, and $\boldsymbol{p} \geqq \mathbf{0}$ is the associated eigenvector.

2.5. Assume that r is the strictly dominant Frobenius root, i.e. if $\rho \in \mathbb{C}$, $\rho \neq r$ is an eigenvalue of A, then $|\rho| < r$. Algebraically, this is equivalent to the fact that $\phi_A(\lambda)$ does not have a factor of the form $\lambda^i - r^i$ $(i > 1)$.

Then, the following statement holds:

For any $\boldsymbol{l}_0 \geqq \mathbf{0}$, there exists an unique number $c_0 > 0$ and a vector $\boldsymbol{y} \geqq \mathbf{0}$ such that

(i) $\boldsymbol{l}_0 = c_0\boldsymbol{p} + \boldsymbol{y}$.

(ii) $\boldsymbol{l}_t = A^t\boldsymbol{l}_0 = c_0 r^t\boldsymbol{p} + A^t\boldsymbol{y}$ such that $r^{-t}(A^t\boldsymbol{y}) \to 0$ (as $t \to \infty$).

We only illustrate this in the case in which $\phi_A(\lambda) = 0$ has n distinct roots (therefore, A is diagonalizable). Let $r_0(=r), r_1, r_2, ..., r_{n-1}$ the eigenvalues of A, and $\boldsymbol{p}_0(=\boldsymbol{p}), \boldsymbol{p}_1, \boldsymbol{p}_2, ..., \boldsymbol{p}_{n-1}$, the corresponding eigenvectors. Then, since $(\boldsymbol{p}_0, \boldsymbol{p}_1, \boldsymbol{p}_2, ..., \boldsymbol{p}_{n-1})$ form a basis of $\mathbb{C}^n$, there exist unique $c_0, c_1, c_2, ..., c_{n-1}$ such that $\boldsymbol{l}_0 = c_0\boldsymbol{p}_0 + c_1\boldsymbol{p}_1 + c_2\boldsymbol{p}_2 + \ldots + c_{n-1}\boldsymbol{p}_{n-1}$. Applying A for t times, we have

$$\boldsymbol{l}_t = A^t\boldsymbol{l}_0 = c_0 r_0^t\boldsymbol{p}_0 + c_1 r_1^t\boldsymbol{p}_1 + c_2 r_2^t\boldsymbol{p}_2 + \cdots + c_{n-1} r_{n-1}^t\boldsymbol{p}_{n-1}.$$

By assumption, $r_0 = r > |r_k|$ (for $k \geqq 1$), $|r_0^{-t} r_k^t\boldsymbol{p}_k| = |(r_k/r_0)^t\boldsymbol{p}_k| \to 0$ (as $t \to \infty$). Thus $c_0 > 0$. Therefore, the statement holds in this case. For the general case where A is not diagonalizable, this statement is shown by using the Jordan canonical form.

2.6. In summary, we have shown that there exists the population $P_S(x,t) = N(t)\cdot C(x)$, where $N(t) = Ar^t$ and $C(x) = Br^{-(x-1)}{}_{x-1}p_0$, such that

$$\lim_{t\to\infty} \frac{l(x,t)}{N(t)} = \frac{P_S(x,t)}{N(t)} = C(x).$$

Here,

$$A = c_0 \sum_{x=0}^{n-1} r^{-x}{}_x p_0 \qquad ; \qquad B = \left(\sum_{x=0}^{n-1} r^{-x}{}_x p_0\right)^{-1}.$$

This result establishes the existence of the stable population in a discrete case.

References

Hoppensteadt, F. C. (1982). *Mathematical methods of population biology*, Cambridge Univ. Press.

United Nations. (1968). *The Concept of a Stable Population.*

Inaba, H. (2002). *Mathematical Models for Demography and Epidemics* [in Japanese], Univ. of Tokyo Press.

Chapter 8

Theory of Lorenz Curves and its Applications to Income Distribution Analysis

Part I. General Theory of Lorenz Curves

1. Introduction

The Lorenz curve is a statistical device used for analysing income distribution. Let $f(t)$ be a probability density function of income t, where t ranges $\alpha \leqq t \leqq \beta$. The Lorenz curve corresponding to this income distribution is given by plotting the points $(x, y) = (x(t), y(t))$, where $x = x(t)$ is the percentage of population who earn at the level equal to or less than t, and $y = y(t)$ is the percentage of the total income earned by this population. In terms of equation,

$$x = x(t) = \int_\alpha^t f(s)ds \qquad ; \qquad y = y(t) = \frac{1}{m}\int_\alpha^t sf(s)ds,$$

where $m = \displaystyle\int_\alpha^\beta tf(t)dt$ is the mean of f.

The above definition gives the parameter representation of the Lorenz curve. To eliminate t, putting $u = F(s)$, where $F(t)$ is the cumulative distribution function of $f(t)$, leads to $du = dF(s) = f(s)ds$ and $s = F^{-1}(u)$. Hence, we have

$$x = F(t) \qquad ; \qquad y = \frac{1}{m}\int_{F(\alpha)}^{F(t)} F^{-1}(u)du = \frac{1}{m}\int_0^x F^{-1}(u)du =: l(x).$$

In Part I, we focus on the theoretical aspect of the Lorenz curve method, with an objective of establishing a mathematical framework for the correspondence between income distributions and their Lorenz curves. Firstly, we will show that the above argument is extended to general statistical distributions defined on non-negative real numbers. Secondly, on the basis of the main theorem we will give a basic formula of the transformation of the Lorenz curves. Thirdly, we will study measures of inequality such as the Gini coefficient. In this paper, we will use freely the standard terminology and basic knowledge of the theory of Lebesgue integration. Two mathematical appendices supplement detailed technical issues. In Part II, we shall provide various examples and applications.

2. Mathematical Preliminaries

In this section we prepare some notions necessary for the formulation of the main Theorem.

Lemma 1 *Let $F(x)$ be a non-decreasing, right-continuous, real-valued function on an open set $U \subset \mathbb{R}$ and let D be an open subset of $F(U) = \{F(x); x \in U\}$. For $x \in D$, define $F^b(x) := \inf\{t; F(t) \geqq x\}$. Then the following statements hold:*

(i) $F^b(x)$ *is non-decreasing on* D.

(ii) $F^b(x)$ *is left-continuous on* D.

(iii) $F(x) \geqq y \Longleftrightarrow F^b(y) \leqq x$.

Proof. (i) Let $x \leqq y$. Since F is non-decreasing, we have $\{t; F(t) \geqq x\} \supset \{t; F(t) \geqq y\}$. Thus, $F^b(x) \leqq F^b(y)$.

(ii) Noting that F^b is non-decreasing, it is sufficient to show that $\sup\limits_{\delta>0} F^b(x-\delta) \geqq F^b(x)$. Put $a = \sup\limits_{\delta>0} F^b(x-\delta)$. Then,

$$
\begin{array}{ll}
 & \forall \varepsilon > 0, \forall \delta > 0 : F(a+\varepsilon) \geqq x - \delta \\
\Longrightarrow & \forall \varepsilon > 0 : F(a+\varepsilon) \geqq x \\
\Longrightarrow & F(a) \geqq x \qquad \text{(right-continuity)} \\
\Longrightarrow & F^b(x) \leqq a.
\end{array}
$$

(iii) We have

$$
\begin{array}{ll}
 & F^b(y) \leqq x \\
\Longleftrightarrow & \inf\{t; F(t) \geqq y\} \leqq x \\
\Longleftrightarrow & \forall \varepsilon > 0 : x + \varepsilon \in \{t; F(t) \geqq y\} \\
\Longleftrightarrow & \forall \varepsilon > 0 : F(x+\varepsilon) \geqq y \\
\Longleftrightarrow & F(x) \geqq y \qquad \text{(right-continuity).} \qquad \text{(Q.E.D.)}
\end{array}
$$

Reciprocally, we have:

Lemma 2 *Let $G(x)$ be a non-decreasing, left-continuous, real-valued function on an open set $V \subset \mathbb{R}$ and let E be an open subset of $G(V) = \{G(x); x \in V\}$. For $x \in E$, define $G^{\#}(x) := \sup\{t; G(t) \leqq x\}$. Then,*

(i) $G^{\#}(x)$ *is non-decreasing on* E.

(ii) $G^{\#}(x)$ *is right-continuous on* E.

(iii) $G(x) \leqq y \iff G^{\#}(y) \geqq x$.

Proof. Similar to Lemma 1.

Lemma 3 *In the same notation as Lemma 1 and 2, we have*

(i) $(F^b)^{\#}(x) = F(x)$ *on* U.

(ii) $(G^b)^{\#}(x) = G(x)$ *on* V.

Consequently, if F is strictly increasing and continuous, then $F^b(x) = F^{\#}(x) = F^{-1}(x)$.

Proof. (i) From Lemma 1 (i) and (ii), $F^b(x)$ is non-decreasing and left-continuous. Hence, from Lemma 1 (iii) and Lemma 2 (iii), $F(x) \geqq y \iff F^b(y) \leqq x \iff (F^b)^{\#}(x) \geqq y$. Thus, $F(x) = (F^b)^{\#}(x)$. (In fact, if there exists some x_0 such that $F(x_0) \neq (F^b)^{\#}(x_0)$, then $y = \frac{1}{2}(F(x_0) + (F^b)^{\#}(x_0))$ will give rise to a contradiction). The proof of (ii) is similar. (Q.E.D.)

Lemma 4 *Let C be the class of function $F : \mathbb{R} \to [0,1]$ which has the following properties:*

(F-i) *If* $x \leqq y$ *then* $F(x) \leqq F(y)$ *(non-decreasing).*

(F-ii) $\lim_{\varepsilon \downarrow 0} F(x+\varepsilon) = F(x)$ *(right-continuous).*

(F-iii) $\lim_{x \to -\infty} F(x) = 0, \ \lim_{x \to \infty} F(x) = 1.$

Then, for $F \in C, X_F(x) = F^b(x)$ is a random variable on a certain probability space whose cumulative distribution function is $F(x)$. Conversely, for any random variable X on some probability space, $F_X(x) = Prob(X \leqq x)$ belongs to C. (See Mathematical Appendix A).

Proof. Since $X_F(x) = F^b(x)$ is non-decreasing, it is measurable. Hence, X_F is a random variable on $((0,1), \mathbf{B_1}(0,1), \lambda_0)$, where $\mathbf{B_1}(0,1)$ denotes the Borel σ-algebra on $(0,1)$; and, λ_0, the Lebesgue measure on $(0,1)$. (For the definition of these terms, see Mathematical Appendix A).

To show that the cumulative distribution function of X_F is identical to $F(x)$, we note from Lemma 1 (iii) that $F^b(\xi) \leqq x \iff F(x) \geqq \xi$. Hence, $Prob(X_F \leqq x) = \lambda_0(0, F(x)] = F(x)$.

The latter part can be verified by using the properties of probability measure. (See Mathematical Appendix A). (Q.E.D.)

In view of the above one-to-one correspondences $F \to X_F$ and $X \to F_X$, we call $F^b(x)(= X_F(x))$ the probability representing function[1] of F.

Notation. Let

$$C_m = \Big\{F \in C;\ E[X_F] = \int_{-\infty}^{\infty} x dF(x) = m\Big\},$$

$$C_m^+ = \{F \in C_m;\ F(x) = 0\ (x < 0)\}.$$

Noting that $C_m \cap C_{m'} = \emptyset$ $(m \neq m')$, we define

$$\mathcal{R} = \bigcup_{m>0} C_m^+ \qquad \text{(disjoint sum)}.$$

Let $\mathcal{L}$ be the class of function $l : [0,1] \to [0,1]$ with the following properties:

(L-i) $l(x)$ is continuous on $[0,1]$; and, $l(0) = 0$ and $l(1) = 1$.

(L-ii) l is convex on $[0,1]$.

(L-iii) $0 \leqq l'_-(x) \leqq l'_+(x)$ (for $0 < x < 1$).

Note. For basic properties of convex functions and the compatibility of conditions (L-ii) and (L-iii), see Mathematical Appendix B.

3. Main Theorem of the Lorenz Mapping

Theorem 1 *(A) There exists a mapping, called the Lorenz mapping, $\varphi : \mathcal{R} \to \mathcal{L}$ that maps each distribution $F \in \mathcal{R}$ to $\varphi_F \in \mathcal{L}$. The graph of φ_F, i.e., $\{(x,y); y = \varphi_F(x), 0 \leqq x \leqq 1\}$ – and by abuse of language the function $y = \varphi_F(x)$ itself – is called the Lorenz curve of distribution F. The Lorenz mapping is continuous in the following sense: Let $\{F_n\} \subset \mathcal{R}$ and $F \in \mathcal{R}$ such that the corresponding random variables $\{X_{F_n}\}$ converge to X_F almost everywhere on $[0,1]$. Then the sequence of functions $\{\varphi_{F_n}\}$ converge uniformly to φ_F on $[0,1]$. In symbol notation, if $X_{F_n} \to X_F$ (a.e.) then $\varphi_{F_n} \rightrightarrows \varphi_F$ on $[0,1]$.*

(B) Conversely, for any Lorenz curve $l \in \mathcal{L}$ and given $m > 0$, there exists a distribution $\theta(l,m) \in C_m^+$ whose Lorenz curve is identical to l, i.e. $\varphi_{\theta(l,m)} = l$.

Therefore, φ is surjective, and the preimage of $l \in \mathcal{L}$ with respect to the mapping φ can be written as $\varphi^{-1}(l) = \bigcup_{m>0} \theta(l,m)$. The Figure in the next page symbolically illustrates the structure of the Lorenz mapping.

[1] This term is due to Moriguti, see the references.

(C) *If two distributions $F_1 \in C_m^+$ and $F_2 \in C_{m'}^+$ correspond to the same Lorenz curve $l \in \mathcal{L}$, i.e. $\varphi_{F_1} = \varphi_{F_2} = l$, then $F_1(mx) = F_2(m'x)$ (or equivalently, $X_{F_1}/m = X_{F_2}/m'$).*

Proof of (A). [Construction of the Lorenz mapping $\varphi : \mathcal{R} \to \mathcal{L}$] Take any $F \in \mathcal{R}$. Then $F \in C_m^+$ for some $m > 0$. From Lemma 1 we can define $F^b(x)$ on $(0,1)$. $F^b(x)$ has the following properties:

(i) $F^b(x)$ is non-decreasing (by Lemma 1 (i)).

(ii) $X_F(x) = F^b(x)$ is the random variable associated with the distribution $F(x)$ (Lemma 4). In particular,
$$\int_0^1 F^b(t)dt = \int_0^\infty x dF(x) = E[X_F] = m.$$

(iii) $F^b(x) \geqq 0$ $(0 < x < 1)$ (Since $F \in C_m^+$, we have $F(x) = 0$ $(x < 0)$. Thus, for $0 < x < 1, \{t;\ F(t) \geqq x\} \subset \{t \geqq 0\}$. Hence, $F^b(x) \geqq 0$).

Now let
$$\varphi_F(x) = \left(\int_0^x F^b(t)dt\right) : \left(\int_0^1 F^b(t)dt\right) = \frac{1}{m}\int_0^x F^b(t)dt.$$

The above φ_F defines a function $\varphi_F \in \mathcal{L}$. In fact, first, it is seen that φ_F is continuous on $[0,1]$ and $\varphi_F(0) = 0$ and $\varphi_F(1) = 1$; second, since $\varphi_F(x)$ is given by an indefinite integral of a non-decreasing function, $\varphi_F(x)$ is convex. (Theorem B.2. in Mathematical Appendix B) ; third, $\varphi_F'(x) = \frac{1}{m}F^b(x) \geqq 0$ on $(0,1)$.

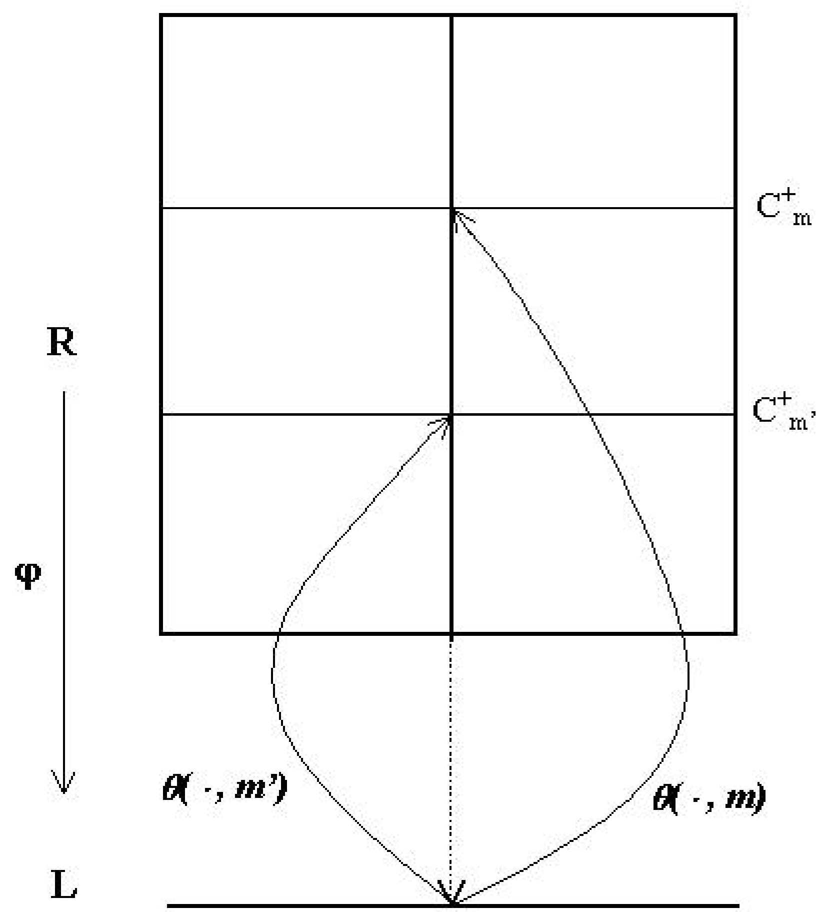
To show the continuity of φ, put $m_n = E[X_{F_n}] = \displaystyle\int_0^1 F_n^b(t)dt < \infty$. Note if $X_{F_n} \to X_F$ (a.e.) on $[0,1]$ then $m_n \to m$. Therefore, for any $0 \leqq x \leqq 1$,
$$|\varphi_{F_n}(x) - \varphi_F(x)| \leqq \int_0^x \left|\frac{F_n^b(t)}{m_n} - \frac{F^b(t)}{m}\right| dt \leqq \int_0^1 \left|\frac{F_n^b(t)}{m_n} - \frac{F^b(t)}{m}\right| dt.$$

Since $F_n^b(t)/m_n \to F^b(t)/m$ (a.e.) and each term is integrable, the RHS converges to zero. (Q.E.D. of (A))

Proof of (B). For $l \in \mathcal{L}$ and $m > 0$, define $G_{l,m} : (0,1) \to \mathbb{R}$ by $G_{l,m}(x) := ml'_-(x)$ (for $0 < x < 1$). Since l is convex, $G_{l,m}$ is well defined and $G_{l,m}(x) \geqq 0$ (L-(iii)) and $G_{l,m}$ is non-decreasing (Theorem B.1.(ii) in Mathematical Appendix B).

Define $\theta(l,m) : \mathbb{R} \to [0,1]$ by $\theta(l,m)(x) := G_{l,m}^{\#}(x)$ (for $x \geqq 0$) and $\theta(l,m)(x) := 0$ (for $x < 0$). Then, $\theta = \theta(l,m)$ satisfies conditions (F-i)-(F-iii). In fact, first, θ is non-decreasing (by Lemma 2 (i)). Second, θ is right-continuous (by Lemma 2 (ii)). Third, $\theta(l,m)(\alpha) = 0,\ \theta(l,m)(\beta) = 1$, where $\alpha = G_{l,m}(0),\ \beta = G_{l,m}(1)$ $(0 \leqq \alpha \leqq \beta \leqq \infty)$; hence, $\displaystyle\lim_{x\to-\infty} \theta(l,m)(x) = 0$ and $\displaystyle\lim_{x\to+\infty} \theta(l,m)(x) = 1$. Therefore, by Lemma 4, $\theta(l,m)$ determines a distribution on $\mathbb{R}$.

The Lorenz Mapping



From Lemma 3, we have $\theta(l,m)^b(x) = (G^{\#}_{l,m})^b(x) = G_{l,m}(x)$ on $(0,1)$. Therefore,

$$E[X_{\theta(l,m)}] = \int_0^1 G_{l,m}(x)dx = m\int_0^1 l'_-(x)dx = m\int_0^1 l'(x)dx = m(l(1)-l(0)) = m$$

(because $l'_-(x) = l'(x)$ a.e.). Hence, $\theta(l,m) \in C^+_m$.

Finally, the Lorenz curve of $\theta = \theta(l,m)$ is

$$\varphi_\theta(x) = \frac{1}{m}\int_0^x G_{l,m}(t)dt = \int_0^x l'_-(t)dt = \int_0^x l'(t)dt = l(x). \qquad \text{(Q.E.D. of (B))}$$

Proof of (C). Suppose $F_1 \in C^+_m,\ F_2 \in C^+_{m'},\ \varphi_{F_1} = \varphi_{F_2} = l$. Then, $F_1(x) = \theta(l,m)(x) = \sup\{t;\ G_{l,m}(t) \leqq x\} = \sup\{t;\ G_{l,1}(t) \leqq x/m\}$. Similarly, $F_2(x) = \theta(l,m')(x) = \sup\{t; G_{l,1}(t) \leqq x/m'\}$. Hence, $F_1(mx) = F_2(m'x)$. (Q.E.D. of (C))

Remark. Let $\mathcal{R}_{AC} = \{F \in \mathcal{R}\ ;\ F$ is absolutely continuous i.e. there exists a density function $f(x) = F'(x)$ a.e.$\}$ and $\mathcal{L}_1 = \{l \in \mathcal{L}\ ;\ l$ is differentiable and l' is continuous$\}$. The Lorenz map φ maps $\mathcal{R}_{AC}$ onto $\mathcal{L}_1$. (See also Theorem A.1. in Mathematical Appendix A).

The Lorenz curve for $F \in \mathcal{R}_{AC}$ is given by: $y = \frac{1}{m}\int_0^x F^b(u)du$. Put $f = F'$ and $x = F(t) = \int_\alpha^t f(s)ds$, then

$$y = \frac{1}{m}\int_0^{F(t)} F^b(u)du = \frac{1}{m}\int_\alpha^t s dF(s) = \frac{1}{m}\int_\alpha^t sf(s)ds,$$

which gives the same expression as we saw in the introduction.

4. Transformation of Lorenz Curves

The following Theorem is useful for studying the effects of changes in income distributions (e.g. by redistribution policies such as taxation and social safety nets).

Theorem 2 *For $F \in C_m^+$, consider the following sequence of mappings:*

$$((0,1), \mathbf{B_1}(0,1), \lambda_0) \xrightarrow{X_F} (\mathbb{R}, \mathbf{B_1}, F) \xrightarrow{g} (\mathbb{R}, \mathbf{B_1}).$$

Here, X_F is the random variable associated with $F(x)$ and g is a Borel measurable function. (For the definition of these terms see Mathematical Appendix A.)

Then $Y = g(X_F)$ is again a random variable on $((0,1), \mathbf{B_1}(0,1), \lambda_0)$, and

(i) If g is left-continuous, then the distribution function of Y is given by $Prob(Y \leqq x) = F(g^\#(x))$ (the definition of $g^\#$ is given in Lemma 2).

(ii) Suppose that (1) g is left-continuous, (2) $(g \circ F^b)(x)$ is non-decreasing, (3) $(g \circ F^b)(x) \geqq 0$ and (4) $\int_\alpha^\beta (g \circ F^b)(t)dt = \mu < \infty$. Then,

$$E[Y] = E[g(X)] = \int_0^1 g(X_F(t))dt = \int_{-\infty}^\infty g(x)dF(x) = \mu,$$

and the Lorenz curve of the distribution of the random variable $Y = g(X_F)$ is given by

$$y = \frac{1}{\mu}\int_0^x g(F^b(t))dt.$$

Proof. (i) From Lemma 2 (iii), $Prob(Y \leqq x) = Prob(g(X) \leqq x) = Prob(X \leqq g^\#(x)) = F(g^\#(x))$. Note that in the proof of Lemma 2 (iii), we did not use the fact that g is non-decreasing.

(ii) For the third equality in the first equation, we refer to Theorem A.3. in Mathematical Appendix A. For the latter statement, we first note that $(F \circ G)^b(x) = G^b(F^b(x))$ if F and G are right-continuous, and $(F \circ G)^\#(x) = G^\#(F^\#(x))$ if F and G are left-continuous (using twice Lemma 1 (iii) and Lemma 2 (iii), respectively). Then, the same argument as the proof of Theorem 1 (B) shows that $(g \circ F^b)^\#(x) = (F^b)^\#(g^\#(x)) = F(g^\#(x)) \in C_\mu^+$ (the above results and Lemma 3 are used). The required statement follows because the probability representing function of Y is $(F \circ g^\#)^b(x) = (g^\#)^b(F^b(x)) = (g \circ F^b)(x)$. (Q.E.D.)

5. Measures of Inequality Associated with Lorenz Curves

The Gini coefficient G is graphically defined as the ratio of the following two areas. The numerator is the area between the line $y = x$ $(0 \leqq x \leqq 1)$ (called the line of complete equality) and the Lorenz curve. The denominator is the triangle area between the line of complete equality and the x-axis, which equals 1/2. Therefore,

$$G = \left(\frac{1}{2} - \int_0^1 l(x)\ dx\right) : \frac{1}{2} = 1 - 2\int_0^1 l(x)dx.$$

The range of Gini coefficient G is $0 \leqq G \leqq 1$. If G is close to 0, the distribution is concentrated; on the contrary, if G is close to 1, the distribution is dispersed.

The maximum discrepancy D is defined as the largest vertical difference between the line of complete equality and the Lorenz curve. In fact, $d(x) = x - l(x) \geqq 0$ is concave on $[0,1]$ and vanishes at $x = 0$ and $x = 1$. Hence, $d(x)$ attains a positive maximum in $(0,1)$ unless $l(x) = x$.

Suppose $l(x)$ is the Lorenz curve of a distribution $F \in C_m^+$. If $l(x)$ is continuously differentiable, the maximum is attained at ξ such that $d'(\xi) = 1 - F^b(\xi)/m = 0$. If F is strictly increasing, then $\xi = F(m)$ and $D = d(F(m)) = F(m) - l(F(m))$.

Proposition 1 *For $F \in C_m$, define*

$$\Delta_0 = \int_{-\infty}^{\infty} |x - m| dF(x) \qquad \textit{(Mean deviation)},$$
$$\Delta_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x - y| dF(x)dF(y) \qquad \textit{(Mean difference)}.$$

Then $\Delta_0 = 2mD$ and $\Delta_1 = 2mG$.

Proof.

$$\begin{aligned}\Delta_0 &= 2\int_{-\infty}^{m}(m - x)dF(x) = 2\left(m\int_{-\infty}^{m} dF(x) - \int_{-\infty}^{m} xdF(x)\right)\\ &= 2\left(mF(m) - \int_0^{F(m)} F^b(t)dt\right) = 2m(F(m) - l(F(m))) = 2mD.\end{aligned}$$

$$\begin{aligned}\Delta_1 &= \int_0^1\int_0^1 |F^b(s) - F^b(t)|\ dsdt = 2\int_0^1 dt \int_0^1 (F^b(t) - F^b(s))ds\\ &= 2\int_0^1 dt(tF^b(t) - ml(t)) = 2m\left([tl(t)]_0^1 - \int_0^1 l(t)dt - \int_0^1 l(t)dt\right)\\ &= 2m\left(1 - 2\int_0^1 l(t)dt\right) = 2mG. \qquad \text{(Q.E.D.)}\end{aligned}$$

6. Concluding Remarks

The author believes that this paper establishes a mathematical basis of the Lorenz curves for a general class of statistical distribution functions[2].

In Part II, we shall study various applications of the general methodology established in this Part.

[2]For instance, the method is applicable for a case such as Cantor's singular distribution - a distribution whose support lies in a measure zero set. The main Theorem ensures the existence of the Lorenz curve of this distribution, but the author does not have any further information on the Lorenz curve of this function.

In practice, it would be sufficient to consider piecewise continuous distribution functions (note that the class of absolutely continuous functions is not large enough to include discrete distributions). In this case, all distributions may be considered to have density functions, if the class of density functions is extended to the delta function.

Mathematical Appendix A - Elements in Probability Theory

This appendix summarises basic concepts and results in probability theory. For further detail, please see references.

Definition A.1. Let Ω be a non-empty set. A family of subset of Ω, denoted by $\mathcal{F} \subset P(\Omega)$, is called *$\sigma$-algebra* if

(S-i) $\Omega \in \mathcal{F}$;

(S-ii) $A \in \mathcal{F} \Longrightarrow A^c \in \mathcal{F}$;

(S-iii) $A_i \in \mathcal{F}\ (i \in \mathbb{N}) \Longrightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

Definition A.2. Let $(\Omega, \mathcal{F})$ be a measurable space. A function P defined on $\mathcal{F}$ is called a probability measure if

(P-i) $P(A) \geqq 0 \qquad (A \in \mathcal{F})$;

(P-ii) $A_i \in \mathcal{F}\ (i \in \mathbb{N}),\ A_i \cap A_j = \emptyset\ (i \neq j) \Longrightarrow P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$;

(P-iii) $P(\Omega) = 1$.

The pair $(\Omega, \mathcal{F}, P)$ is called a probability space. We call Ω a sample space; $A \in \mathcal{F}$, an event; $P(A)$, the probability of the event A, respectively.

Definition A.3. Let $(\Omega, \mathcal{F}, P)$ be a probability space. A real valued $\mathcal{F}$-measurable function $X : (\Omega, \mathcal{F}, P) \to (\mathbb{R}, \mathbf{B_1})$ is called a random variable.

For $A \in \mathbf{B_1}$, put $\mu(A) = P(X^{-1}(A))$, then $(\mathbb{R}, \mathbf{B_1}, \mu)$ is a probability space; and, $Prob(X \in A) = \mu(A)$ is called the probability that X belongs to A.

Function $F(x) = \mu(-\infty, x] = \text{Prob}(X \leqq x)$ (put $A = (-\infty, x]$) is called the (cumulative) distribution function of random variable X.

Theorem A.1. *Every distribution function F has a decomposition $F = F_{AC} + F_{CS} + F_D$, where F_{AC} : absolutely continuous part, F_{CS} : continuous singular part, and F_D : discrete part.*

If $F_{CS} = F_D = 0$, then there exists a measurable function $f \geqq 0$ such that

$$F(x) - F(a) = \int_a^x f(t)\ dt.$$

Here, $f(x)$ is called the probability density function of random variable X.

Theorem A.2. *Suppose a function $F : \mathbb{R} \to [0,1]$ satisfies (i) non-decreasing ; (ii) right-continuous ; (iii) $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to \infty} F(x) = 1$. Then there exists a unique probability measure P on $(\mathbb{R}, \mathbf{B_1})$ such that $P(-\infty, x] = F(x)$. This measure is called the Lebesgue-Stieltjes measure with respect to $F(x)$.*

Theorem A.3. *Consider the following sequence of mappings between two probability spaces and a measurable space:*

$$(X, \mathbf{L}, \lambda) \xrightarrow{\varphi} (Y, \mathbf{M}, \mu) \xrightarrow{g} (Z, \mathbf{N})$$

where φ is $(\mathbf{L}, \mathbf{M})$-measurable, i.e., $A \in \mathbf{M} \Longrightarrow \varphi^{-1}(A) \in \mathbf{L}$;

g is $(\mathbf{M}, \mathbf{N})$-measurable, i.e., $B \in \mathbf{N} \Longrightarrow g^{-1}(B) \in \mathbf{M}$; and,

μ is the induced measure of λ by φ on $(Y, \mathbf{M})$, i.e., $\mu(A) = \lambda(\varphi^{-1}(A))$ (for $A \in \mathbf{M}$).

Then, $g \circ \varphi$ is $(\mathbf{L}, \mathbf{N})$-measurable and for $A \in \mathbf{M}$ it follows that

$$\int_{\varphi^{-1}(A)} g(\varphi(t)) d\lambda(t) = \int_A g(s) d\mu(s) \leqq \infty.$$

Mathematical Appendix B - Convex Functions

Definition B.1. - A real-valued function f defined on an interval $I = [a, b]$ is called convex on I if for $x, y \in I$ and $0 \leqq t \leqq 1 : f((1-t)x + ty) \leqq (1-t)f(x) + tf(y)$.

Theorem B.1. *A convex function f has the following properties:*

(i) *f is continuous on (a, b). Further, if f is bounded on (a, b), thus f can be made continuous on $[a, b]$ (by changing the values of end points, if necessary).*

(ii) *f has both right and left derivatives for each $x \in (a, b)$ and $f'_-(x) \leqq f'_+(x)$. In addition, for $x < y$, $x, y \in (a, b) : f'_+(x) \leqq f'_-(y)$. At the end points, if there exist A and B such that $A \leqq f'_-(x) \leqq f'_+(x) \leqq B$ (for $x \in (a, b)$), then $f'_+(a)$ and $f'_-(b)$ exist and $A \leqq f'_+(a) \leqq f'_-(b) \leqq B$.*

(iii) *The set of indifferentiable points of f in (a, b) is countable (thus, its measure is zero). Therefore, f is differentiable almost everywhere in (a, b) and $f'(x) = f'_-(x) = f'_+(x)$ (a.e.).*

Lemma. *In the above notation, for $x < y$, $x, y \in I$, put $S(x, y) = \dfrac{f(y) - f(x)}{y - x} = S(y, x)$.*

Then for $x < z < y$, it follows that $S(x, z) \leqq S(x, y) \leqq S(z, y)$ i.e.

$$\frac{f(z) - f(x)}{z - x} \leqq \frac{f(y) - f(x)}{y - x} \leqq \frac{f(y) - f(z)}{y - z}.$$

(Note: Consider also the geometrical meaning of the above inequalities.)

Proof of Lemma. The required inequalities are equivalent to $(y - x)f(z) \leqq (y - z)f(x) + (z - x)f(y)$. To show this, put $z = (1 - t)x + ty$, where $t = \dfrac{z - x}{y - x} \in [0, 1]$. Thus $f(z) \leqq (1 - t)f(x) + tf(y)$. Hence,

$$f(z) \leqq \frac{y - z}{y - x} f(x) + \frac{z - x}{y - x} f(y). \qquad \text{(Q.E.D. of lemma)}$$

Proof of Theorem B.1. (i) For a fixed $x \in (a, b)$, there exists $h > 0$ such that $[x - h, x + h] \subset (a, b)$. On the one hand, for $0 < t < 1$, $f((1 - t)x + t(x + h)) \leqq (1 - t)f(x) + tf(x + h)$. Hence, $f(x + th) - f(x) \leqq t(f(x + h) - f(x))$. On the other hand, as $0 < 1/(1 + t) < 1$, we get

$$f\left(\left(1 - \frac{1}{1+t}\right)(x - h) + \frac{1}{1+t}(x + th)\right) \leqq \left(1 - \frac{1}{1+t}\right) f(x - h) + \frac{1}{1+t} f(x + th).$$

Thus, $t(f(x) - f(x - h)) \leqq f(x + th) - f(x)$. Combining the above two results, we have $t(f(x) - f(x - h)) \leqq f(x + th) - f(x) \leqq t(f(x + h) - f(x))$.

If t tends to zero, then both ends terms will tend to zero. Thus $\lim_{t\to 0} f(x+th) = f(x)$.

(ii) Take sufficiently small $\varepsilon > 0$ such that $0 < \varepsilon < h$. Then, from Lemma, we have $S(x-h,x) \leqq S(x-\varepsilon,x) \leqq S(x,x+\varepsilon) \leqq S(x,x+h)$.

If $\varepsilon > \varepsilon' > 0$ then $S(x-\varepsilon,x) \leqq S(x-\varepsilon',x) \leqq S(x,x+\varepsilon') \leqq S(x,x+\varepsilon)$. Hence, both $f'_-(x) = \lim_{\varepsilon\downarrow 0}(f(x)-f(x-\varepsilon))/\varepsilon = \sup_{\varepsilon>0} S(x-\varepsilon,x)$ and $f'_+(x) = \lim_{\varepsilon\downarrow 0}(f(x+\varepsilon)-f(x))/\varepsilon = \inf_{\varepsilon>0} S(x,x+\varepsilon)$ exist, and $f'_-(x) \leqq f'_+(x)$.

If $x < y$, then there exists a sufficiently small $\varepsilon > 0$ such that $x < x+\varepsilon < y-\varepsilon < y$. Thus, from Lemma $S(x,x+\varepsilon) \leqq S(x,y-\varepsilon) \leqq S(y-\varepsilon,y)$. Taking $\varepsilon \downarrow 0$, we have $f'_+(x) \leqq f'_-(y)$.

(iii) Put $j(x) = f'_+(x) - f'_-(x) \geqq 0$. Remark first that $j(x) = 0$ is equivalent to that f is differentiable at x, and second, for $a < x_1 < x_2 < \cdots < x_p < b : 0 \leqq j(x_1) + j(x_2) + \cdots + j(x_p) \leqq f'_+(x_p) - f'_-(x_1)$. From the first remark, the set of indifferentiable points is $K = \{x \in (a,b); j(x) > 0\} = \bigcup_{k=1}^{\infty}\bigcup_{n=1}^{\infty} N_{k,n}$, where $N_{k,n} = \{x \in [a+\frac{1}{n}, b-\frac{1}{n}]; j(x) \geqq 1/k\}$. From the second remark $\#N_{k,n}/k \leqq f'_+(b-\frac{1}{n}) - f'_-(a+\frac{1}{n})$. Thus, $\#N_{k,n} \leqq k\cdot[f'_+(b-\frac{1}{n}) - f'_-(a+\frac{1}{n})] < \infty$. Hence, K is a countable set. (Q.E.D.)

Theorem B.2. *A function f is convex on $[0,1]$ $\iff$ There exists a non-decreasing function ρ on $(0,1)$ such that*

$$f(x) = f(0) + \int_0^x \rho(t)dt.$$

Proof. Suppose f is convex, then $f'_-(x)$ exists for $x \in (0,1)$. Since $f'_-(x)$ is non-decreasing and $f'_-(x) = f'_+(x) = f'(x)$ (a.e.). Thus putting $\rho(x) = f'_-(x)$ will yield

$$f(x) = f(0) + \int_0^x f'(t)dt = f(0) + \int_0^x f'(t)dt = f(0) + \int_0^x \rho(t)dt.$$

Conversely, if f is given in the above integral, for $x, y, t \in [0,1]$ and $x < y$,

$$\begin{aligned} &(1-t)f(x) + tf(y) - f((1-t)x+ty) \\ =&(1-t)\int_0^x \rho(s)ds + t\int_0^y \rho(s)ds - \int_0^{(1-t)x+ty} \rho(s)ds \\ =&t\int_x^y \rho(s)ds - \int_x^{(1-t)x+ty} \rho(s)ds \end{aligned}$$

[In the second integral, change variable by $s = t(u-x)+x$]

$$=t\left(\int_x^y \rho(u)du - \int_x^y \rho(tu+(1-t)x)du\right) \geqq 0. \qquad \text{(Note } x \leqq u \leqq y.\text{)}$$

Hence, f is convex. (Q.E.D.)

Part II. Examples and Applications

1. Introduction and Summary of Part I

This Part is devoted to applications of the general theory established in Part I. First, we calculate Lorenz curves and Gini coefficients for various income distributions. Second, we consider some notions on the comparison of income distributions. Third, we study the effects of redistribution policies on the Lorenz curve.

For convenience, we summarise here the algorithm on the calculation of the Lorenz curve and Gini coefficient for a given distribution function.

(1) Given a cumulative distribution function: $F(x)$.

(2) Find the probability representing function of F: $F^b(t) = \inf\{x; F(x) \geqq t\}$.

(3) Find the primitive of $F^b(t) : I_F(x) = \int_0^x F^b(t)dt$.

(4) The mean is given by: $m = I_F(1)$.

(5) The Lorenz curve is given by: $y = l(x) = I_F(x)/m$.

(6) The Gini coefficient is calculated as: $G = 1 - 2\int_0^1 l(x)dx = 1 - \frac{2}{m}\int_0^1 I_F(x)dx$.

2. Applications to Typical Income Distributions

For brevity, hereafter, c.d.f. stands for the cumulative distribution function, and p.r.f., for the probability representing function.

Example 2.1. (Equal distribution or one-point distribution)

c.d.f. $F(x) = \begin{cases} 0 & (0 \leqq x < m) \\ 1 & (m \leqq x) \end{cases}$

p.r.f. $F^b(t) = m \qquad (0 < t < 1)$

Mean: m

Lorenz curve: $l(x) = x \qquad (0 \leqq x \leqq 1)$

Gini coefficient: $G = 0$

Remark. This example corresponds to an extreme case in which each member of a group earns the same amount of income. For this reason, this Lorenz curve is called the line of complete equality.

Example 2.2. (Two points distribution)

c.d.f. $$F(x) = \begin{cases} 0 & (0 \leqq x < a) \\ \theta & (a \leqq x < b) \\ 1 & (b \leqq x) \end{cases}$$

p.r.f. $$F^b(t) = \begin{cases} a & (0 < t \leqq \theta) \\ b & (\theta < x < 1) \end{cases}$$

Mean: $a\theta + b(1-\theta)$

Lorenz curve: $$l(x) = \begin{cases} \frac{ax}{a\theta+b(1-\theta)} & (0 \leqq x \leqq \theta) \\ \frac{a\theta+b(x-\theta)}{a\theta+b(1-\theta)} & (\theta \leqq x \leqq 1) \end{cases}$$

Gini coefficient: $$G = \theta - \frac{a\theta}{a\theta + b(1-\theta)}$$

Remarks. (i) This example corresponds to the case where the total population consists of two income groups - the high income group and the low income group.

(ii) The Lorenz curve consists of two segments linking $O = (0,0), A = (\theta, a\theta/[a\theta + b(1-\theta)])$ and $I = (1,1)$.

(iii) If $a = 0$, then $A = (\theta, 0)$ and $G = \theta$. In addition, if we take the limit $\theta \to 1$, the Lorenz curve will tend to the lines that link $O = (0,0), A' = (1,0)$ and $I = (1,1)$, and G will tend to 1. (Note that the limit curve does not belong to $\mathcal{L}$). This corresponds to the other extreme case where the inequality of income is very large.

(iv) By induction, in the case of n points distribution at a_k with weight $\theta_k (1 \leqq k \leqq n)$, where $\sum \theta_k = 1$, the Lorenz curve consists of n segments $A_{k-1}A_k (1 \leqq k \leqq n)$. Here,

$$A_k = (x_k, y_k), \text{ where } x_k = \sum_{i=1}^{k} \theta_i \text{ and } y_k = \left(\sum_{i=1}^{k} a_i\theta_i \right) / \left(\sum_{i=1}^{n} a_i\theta_i \right).$$

(Note that $A_0 = O = (0,0)$, $A_n = I = (1,1)$).

The Gini coefficient of this distribution is calculated as: $G = 1 - \sum_{k=1}^{n} (y_{k-1} + y_k)\theta_k$.

Example 2.3. (Uniform distribution)

c.d.f. $$F(x) = \begin{cases} 0 & (x < a) \\ (x-a)/\theta & (a \leqq x < a + \theta) \\ 1 & (x \geqq a + \theta) \end{cases}$$

p.r.f. $F^b(t) = a + \theta t$

Mean: $a + \theta/2 = m$

Lorenz curve: $$l(x) = \frac{x(\theta x + 2a)}{2a + \theta} = x\left(\frac{\theta(x-1)}{2m} + 1\right) \qquad (0 < t < 1)$$

Gini coefficient: $G = \dfrac{\theta}{6a + 3\theta} = \dfrac{\theta}{6m}$

Remarks. (i) For a fixed $m > 0$, if the width θ tends to zero, then it converges to one point distribution. Thus $l(x) = x$ and $G = 0$.

(ii) For a fixed $a > 0$, if θ tends to infinity, then $l(x) = x^2$ and $G = 1/3$.

Example 2.4.

c.d.f. $F(x) = \left(\dfrac{x-a}{\theta}\right)^c \qquad (a \leqq x \leqq a+\theta,\ c > 0)$

p.r.f. $F^b(t) = a + \theta t^{1/c}$

Mean: $a + c\theta/(c+1)$

Lorenz curve: $l(x) = \dfrac{c+1}{a(c+1)+c\theta}\left(ax + \dfrac{c\theta}{c+1}x^{\frac{c+1}{c}}\right)$

Gini coefficient: $G = 1 - \dfrac{1}{2c+1} \cdot \dfrac{a(c+1)(2c+1) + 2c^2\theta}{a(c+1)+c\theta}$

Remarks. (i) This example includes Example 2.3 as a special case of $c = 1$.

(ii) For a fixed $a > 0$, if $\theta \to 0$ then $l(x) = x$ and $G = 0$.

(iii) For a fixed $a > 0$, if $\theta \to \infty$, then $l(x) = x^{\frac{c+1}{c}}$ and $G = \frac{1}{2c+1}$.

Example 2.5. (Pareto distribution)

c.d.f. $F(x) = 1 - (a/x)^c \qquad (x > a,\ c > 1)$

p.r.f. $F^b(t) = a(1-t)^{-1/c}$

Mean: $ca/(c-1)$

Lorenz curve: $l(x) = 1 - (1-x)^{\frac{c-1}{c}}$

Gini coefficient: $G = 1/(2c-1)$

Example 2.6. (Inverse Beta distribution)

In this case, the distribution is given by p.r.f.

p.r.f. $F^b(t) = t^{\alpha-1}(1-t)^{\beta-1} \qquad (\alpha \geqq 1,\, 0 < \beta \leqq 1)$

Mean: $B(\alpha,\beta) = \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$

Lorenz curve: $l(x) = \dfrac{1}{B(\alpha,\beta)} \displaystyle\int_0^x t^{\alpha-1}(1-t)^{\beta-1}dt$

Gini coefficient: $G = (\alpha - \beta)/(\alpha + \beta)$

Remarks. (i) This distribution may be regarded as an interpolation between the cases in Examples 2.4 and 2.5. (Note that $\beta = 1$ will lead to Example 2.4 and $\alpha = 1$ will lead to Example 2.5).

(ii) To calculate the Gini coefficient we note that

$$\begin{aligned}\int_0^1 l(x)dx &= [xl(x)]_0^1 - \int_0^1 xl'(x)dx = 1 - \frac{1}{B(\alpha,\beta)}\int_0^1 x^\alpha(1-x)^{\beta-1}dx \\ &= 1 - \frac{B(\alpha+1,\beta)}{B(\alpha,\beta)} = 1 - \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} = 1 - \frac{\alpha}{\alpha+\beta} = \frac{\beta}{\alpha+\beta}.\end{aligned}$$

Example 2.7. (Lognormal distribution)

c.d.f. $F(x) = \Phi\left(\frac{\log x - m}{\sigma}\right), \qquad (x > 0)$

where Φ is the c.d.f. of the standard normal distribution $N(0,1)$: $\Phi(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^m e^{-\frac{t^2}{2}}dt$ and $m > 0, \sigma > 0$.

p.r.f. $F^b(t) = \exp(m + \sigma\Phi^{-1}(t))$

Mean: $\exp(m + \sigma^2/2)$

Variance: $\exp(2m + \sigma^2)(\exp(\sigma^2) - 1)$

Lorenz curve(parameter representation): $x = \Phi(t)$ and $y = \Phi(t - s)$ $(-\infty \leqq t \leqq \infty)$

Gini coefficient: $G = 2\Phi\left(\frac{\sigma}{\sqrt{2}}\right) - 1 = \frac{1}{\sqrt{2\pi}}\int_{-\sigma/\sqrt{2}}^{\sigma/\sqrt{2}} e^{-\frac{x^2}{2}}dx$

Remark. For detail, see Chapter 9: "Note on lognormal and multivariate normal distributions".

Example 2.8. (Gamma distribution)

p.d.f. For $\alpha, \beta > 0$,

$$f(x;\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}, \qquad x \geqq 0.$$

Mean: α/β

Variance: α/β^2

Lorenz curve (parameter representation):

$$x(t) = \frac{\gamma(\alpha,\beta t)}{\Gamma(\alpha)}, \qquad y(t) = \frac{\gamma(\alpha+1,\beta t)}{\Gamma(\alpha+1)}, \qquad (0 \leqq t \leqq \infty)$$

where $\gamma(\alpha, t)$ is the (lower) incomplete gamma function defined by $\gamma(\alpha, t) = \int_0^t s^{\alpha-1} e^{-s}\, ds$. We have used the formula: $\Gamma(z+1) = z\Gamma(z)$. Note that the Lorentz curve does not depend on β. Therefore we hereinafter assume that $\beta = 1$ without any loss of generality.

Gini coefficient:

$$G = \frac{\Gamma(2\alpha+1)}{2^{2\alpha}\Gamma(\alpha+1)^2} = \frac{1}{2^{2\alpha-1}\alpha B(\alpha,\alpha)} = \frac{\Gamma(\alpha+\frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha+1)}.$$

The derivation of the Gini coefficient is as follows. We first note that $x(t) = y(t) + y'(t)$. In fact, by integration by parts, we have

$$\begin{aligned} y(t) &= \frac{1}{\Gamma(\alpha+1)} \int_0^t s^\alpha e^{-s} ds = \frac{1}{\Gamma(\alpha+1)} \left(-s^{-\alpha} e^{-s} \Big|_0^t + \int_0^t \alpha s^{\alpha-1} e^{-s} ds \right) \\ &= \frac{1}{\Gamma(\alpha+1)} (-t^{-\alpha} e^{-t}) + \frac{\gamma(\alpha,t)}{\Gamma(\alpha)} = -y'(t) + x(t). \end{aligned}$$

By using this result, we calculate the Gini coefficient as follows:

$$\begin{aligned} G &= 1 - 2\int_0^1 y(t)dx(t) = 1 - 2\int_0^\infty y(t)x'(t)dt = 1 - 2\int_0^\infty y(t)\left(y'(t) + y''(t)\right) dt \\ &= -2\int_0^\infty y(t)y''(t)dt \qquad (\because \int_0^\infty y(t)y'(t)dt = \int_0^1 ydy = \frac{1}{2}) \\ &= -2\left(yy'\Big|_0^\infty - \int_0^\infty y'(t)^2 dt \right) = 2\int_0^\infty y'(t)^2 dt \qquad (\because yy'\Big|_0^\infty = 0) \\ &= \frac{1}{\Gamma(\alpha+1)^2} \int_0^\infty t^{2\alpha} e^{-2t} 2dt = \frac{1}{2^{2\alpha}\Gamma(\alpha+1)^2} \int_0^\infty s^{2\alpha} e^{-s} ds = \frac{\Gamma(2\alpha+1)}{2^{2\alpha}\Gamma(\alpha+1)^2}. \end{aligned}$$

The other expressions follow from formulae $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, $B(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$, and the double argument formula (putting $z = a + \frac{1}{2}$):

$$2^{2z-1}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = \sqrt{\pi}\Gamma(2z),$$

which is a special case ($n = 2$) of the following formula:

$$\prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}-nz} \Gamma(nz).$$

3. Comparison of Income Distributions

Recall the definition of $\mathcal{R} = \bigcup_{m>0} C_m^+$ where C_m^+ denotes the set of distribution functions with a positive support and the mean $m > 0$.

In $\mathcal{R}$, we can introduce the following orderings:

Definition 3.1. Let $F, G \in \mathcal{R}$ and let $l_F(x)$ and $l_G(x)$ be their Lorenz curves.

(i) $L(F) \geqq L(G) \iff l_F(x) \geqq l_G(x)$ (for $0 \leqq x \leqq 1$) [Note that this is a partial ordering - if two Lorenz curves intersect, they are not comparable in this sense].

(ii) Put Gini(F) = (Gini coefficient of F). Then Gini$(F) \geqq$ Gini(G) makes sense as usual order in real number.

(iii) Fix a concave welfare density function $U(x) \geqq 0$, with $U'(x) \geqq 0$ and $U''(x) \leqq 0$. Define

$$W(F) = \int_0^\infty U(x)dF(x)$$

(this integral is assumed always finite), then $W(F) \geqq W(G)$ makes sense in a similar way to (ii). $W(F)$ may be interpreted as the average utility associated with the income distribution $F(x)$. The concavity of the welfare density function reflects the diminishing marginal utility of income.

Proposition 3.2.

(i) For $F, G \in \mathcal{R}$, if $L(F) \geqq L(G)$, then Gini$(F) \leqq$ Gini(G).

(ii) (Atkinson) Let $F, G \in C_m^+$ and F and G are continuous and strictly increasing. If $L(F) \geqq L(G)$, then $W(F) \geqq W(G)$ (with respect to a positive, concave welfare density function $U \geqq 0, U' \geqq 0$ and $U'' \leqq 0$).

Proof. (i) is graphically evident. To show (ii),

$$\begin{aligned} W(F) - W(G) &= \int_0^\infty U(dF - dG) = [U(F-G)]_0^\infty - \int_0^\infty U'(F-G)dx = -\int_0^\infty U'(F-G)dx \\ &= [U'(J_F - J_G)]_0^\infty + \int_0^\infty U''(J_F - J_G)dx = \int_0^\infty U''(J_F - J_G)dx, \end{aligned}$$

where $J_F(x) = \int_0^x F(t)dt$. It is assumed that $U'(x)$ decreases rapidly so that $\lim_{x\to\infty} U'(x)J_F(x) = \lim_{x\to\infty} U'(x)J_G(x) = 0$.

We claim that $J_F(x) - J_G(x) \leqq 0$. Note that

$$ml_F(F(t)) = \int_0^{F(t)} F^b(u)du = \int_0^t sdF(s) = tF(t) - \int_0^t F(s)ds = tF(t) - J_F(t).$$

Thus,

$$\begin{aligned} J_F(t) - J_G(t) &= t[F(t) - G(t)] - m[l_F(F(t)) - l_G(G(t))] \\ &= t[F(t) - G(t)] - m[l_G(F(t)) - l_G(G(t))] - m[l_F(F(t)) - l_G(F(t))] \\ &= t[F(t) - G(t)] - \int_{G(t)}^{F(t)} G^{-1}(u)du - m[l_F(F(t)) - l_G(F(t))]. \end{aligned}$$

By assumption, the last term is $\leqq 0$. By the mean-value theorem of the definite integral, the first two terms are written by:

$$\begin{aligned} t[F(t)-G(t)] - \int_{G(t)}^{F(t)} G^{-1}(u)du &= t[F(t)-G(t)] - G^{-1}(\theta)[F(t)-G(t)] \\ &= (t - G^{-1}(\theta))[F(t)-G(t)]. \end{aligned}$$

Here, θ is between $F(t)$ and $G(t)$. Therefore, if $G(t) \leqq F(t)$ then $G(t) \leqq \theta$; similarly, if $F(t) \leqq G(t)$ then $\theta \leqq G(t)$. Thus, the above expression is also $\leqq 0$. This completes the proof. (Q.E.D.)

Remark. It should be noted that the above result (ii) does not necessarily imply that an extensive redistribution policy would increase the welfare of an economy (Ultimately, the equal distribution will maximize the welfare). For this argument, at least the following points should be examined: (a) The cost (administrative as well as social) associated with income redistribution, (b) This argument is static in the sense that it concerns only the redistribution of the current income. The impact on future income growth is not taken into account.

Examples 3.3.

(i) Let $U(m, \theta)$ denote the uniform distribution with mean m and width θ. Then, if $\theta \leqq \theta'$ then $L(U(m,\theta)) \geqq L(U(m,\theta'))$.

(ii) Let $P(c)$ denote the Pareto distribution with shape parameter c (regardless of $a > 0$). Then, if $c \leqq c'$ then $L(P(c)) \leqq L(P(c'))$.

(iii) Let $\Lambda(\sigma)$ denote the Lognormal distribution with parameter σ (regardless of m). Then, if $\sigma \leqq \sigma'$ then $L(\Lambda(\sigma)) \geqq L(\Lambda(\sigma'))$

4. Effects of Redistribution Policies

Definition 4.1. Consider a mapping $T : \mathcal{R} \to \mathcal{R}$. For $F \in \mathcal{R}$, $F \in C_m^+$ for some $m > 0$. Then $G = T(F)$ is called a redistribution of F if $G \in C_m^+$. Let $F^b(t)$ and $G^b(t)$ be the probability representing functions of F and G, respectively. For $F \in C_m^+$, $G = T(F)$ is called an income net of taxation, if $G^b(t) \leqq F^b(t)$ $(0 < t < 1)$. By this definition, $G \in C_{m'}^+$, where $m' \leqq m$. We call $t = m - m' \geqq 0$ the average tax.

In most cases, we consider T in the form of the transformation of distributions defined in Theorem 2 of Part I.

In order to show the effects of redistribution policies on Lorenz curves, we take a simple example consisting of three income groups, namely the high income group, the medium income group, and the low income group.

Example 4.2. (Reference distribution)

Fix $0 < a < b < c$, and put

p.r.f.

$$X(t) = \begin{cases} a & (0 < t \leqq 1/3) & \text{:Low income group} \\ b & (1/3 < t \leqq 2/3) & \text{:Medium income group} \\ c & (2/3 < t < 1) & \text{:High income group} \end{cases}$$

Mean $= (a+b+c)/3 = m$.

Lorenz curve: $L(X) = OABI$, where $O = (0,0)$, $A = (1/3, a/(3m))$, $B = (1/3, (a+b)/(3m))$, $I = (1,1)$.

Example 4.3. (Effect of an income transfer)

Consider an income transfer of r $(0 \leqq r \leqq \min(c-b, b-a))$ from the high income group to the low income group, then

p.r.f.

$$Y(t) = \begin{cases} a+r & (0 < t \leqq 1/3) & \text{:Low} \\ b & (1/3 < t \leqq 2/3) & \text{:Medium} \\ c-r & (2/3 < t < 1) & \text{:High} \end{cases}$$

Figure 1-a shows the change in the p.r.f.. The mean pf $Y = (a+b+c)/3$ therefore the income transfer $X \to Y$ is a redistribution.

Lorenz curve: $L(Y) = OA'B'I$, where $A' = (1/3, (a+r)/(3m), B' = (2/3, (a+b+r)/(3m))$. Figure 2-a shows the effect on the Lorenz curve. Since $r > 0$, the middle segment AB will shift upwards to $A'B'$ and $L(X) \leqq L(Y)$. Hence, $\text{Gini}(X) \geqq \text{Gini}(Y)$ and $W(X) \leqq W(Y)$.

Further, if a cost for the transfer $g < r$ is considered, due to the loss of income, mean $= m - g/3$ and the Lorenz curve $= OA_cB_cI$, where $A_c = (1/3, (a+r-g)/(3m-g))$ and $B_c = (2/3, (a+b+r-g)/(3m-g))$. Since $(a+r-g)/(3m-g) > a/(3m)$ and so on, the resulting Lorenz curve will shift upwards. This implies that even if there is a loss of income transfer, so far as the cost is met by the high income group, the resulting distribution will still be more equal.

Example 4.4. (Flat amount taxation)

For the same distribution X in the previous example, consider a flat amount taxation $0 \leqq s \leqq a$.

p.r.f.

$$Z(t) = \begin{cases} a-s & (0 < t \leqq 1/3) \\ b-s & (1/3 < t \leqq 2/3) \\ c-s & (2/3 < t < 1) \end{cases}$$

(See Figure 1-b)

Mean $= (a+b+c)/3 - s$. Therefore $X \to Z$ is a taxation with average tax s.

Figure 1-a

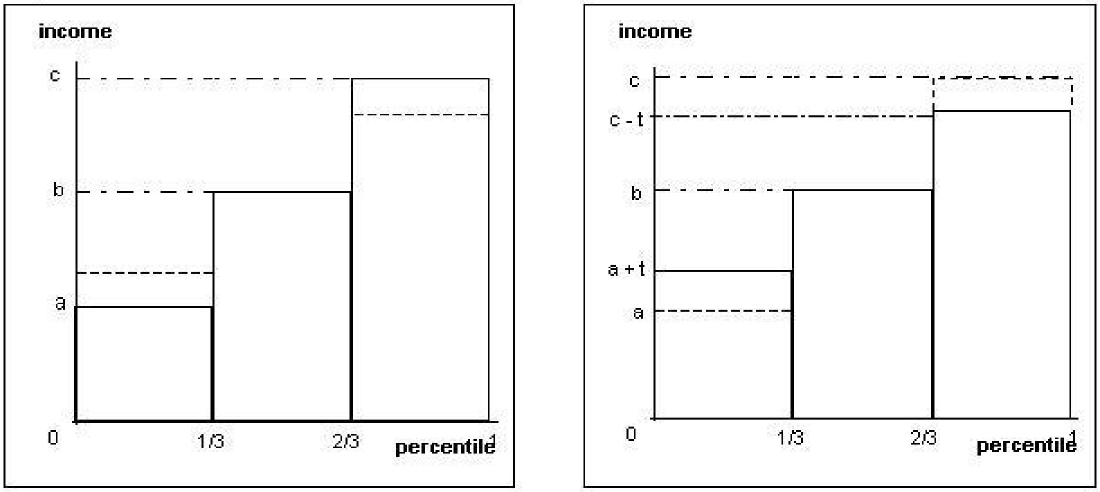


Figure 1-b

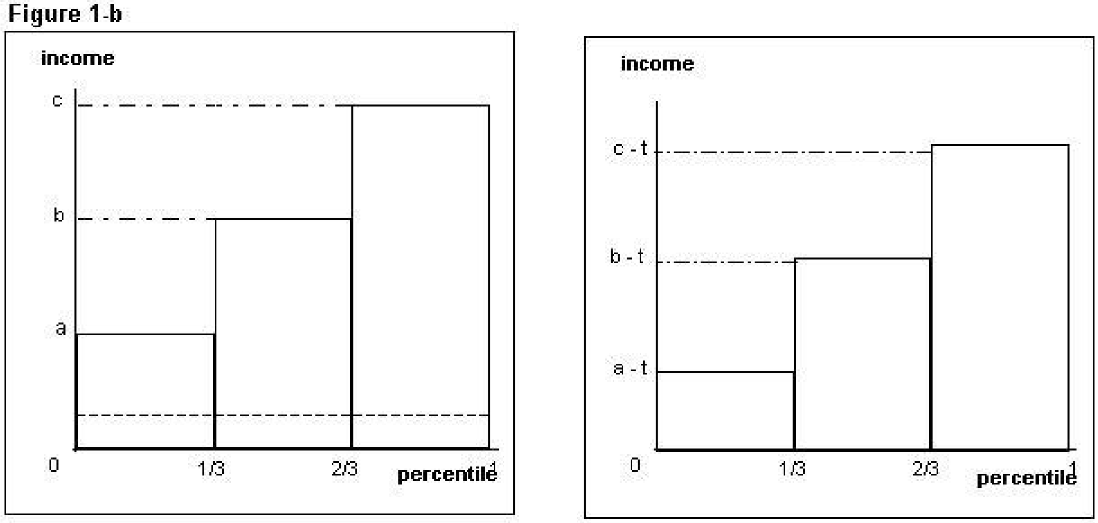


Figure 1-c

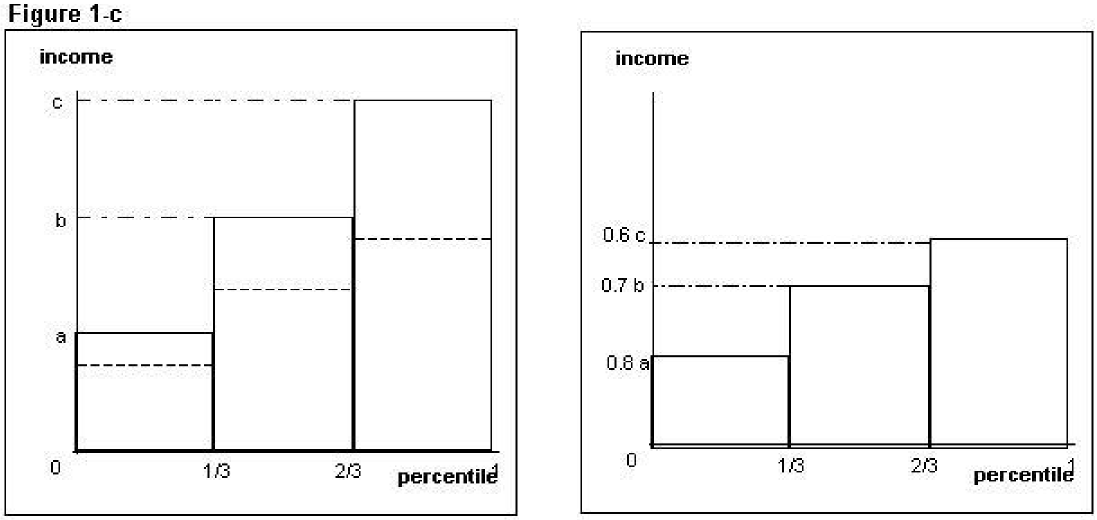


Figure 2-a: Effect of Income Transfer

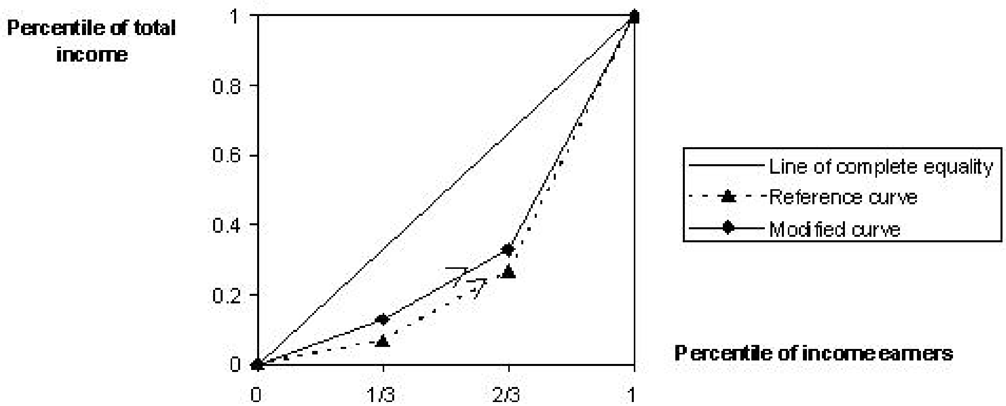


Figure 2-b: Effect of Fixed Taxation

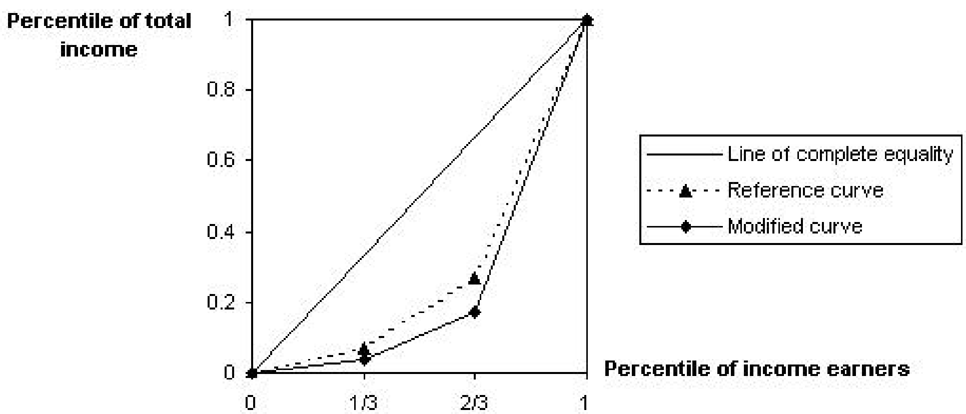
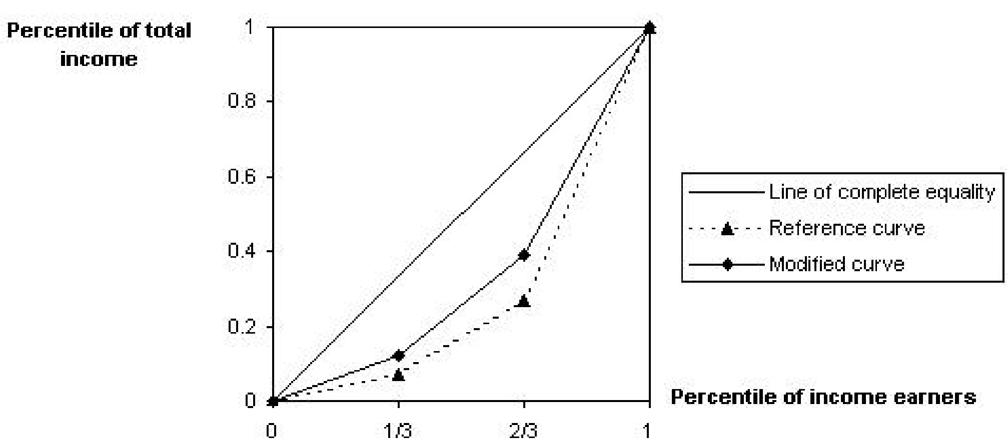


Figure 2-c: Effect of Progressive Taxation



Lorenz curve $L(Z) = OA''B''I$, where, $A'' = (1/3, (a-s)/(3m-3s))$, $B'' = (2/3, (a+b-2s)/(3m-3s))$. A simple calculation shows that $L(Z) \leqq L(X)$. Hence, $\mathrm{Gini}(Z) \geqq \mathrm{Gini}(X)$ and $W(Z) \leqq W(X)$ (See Figure 2-b).

Example 4.5. (Progressive tax rates)

In the same distribution consider the following tax rates: 20% for the low income group, 30% for the medium income group, 40% for the high income group. (assume $0 < 0.8a < 0.7b < 0.6c$)

p.r.f.

$$V(t) = \begin{cases} 0.8a & (0 < t \leqq 1/3) \\ 0.7b & (1/3 < t \leqq 2/3) \\ 0.6c & (2/3 < t < 1) \end{cases}$$

(See Figure 1-c)

$$\text{Mean} = \frac{1}{3}(0.8a + 0.7b + 0.6c) = \frac{1}{3}(a+b+c) - \frac{1}{3}(0.2a + 0.3b + 0.4c) = m - p.$$

Lorenz curve $L(V) = OA'''B'''I$, where $A''' = (1/3, 0.8a/(3m-3p))$, $B''' = (2/3, (0.8a + 0.7b)/(3m-3p))$. A simple calculation shows that $L(V) \geqq L(X)$. Hence, $\mathrm{Gini}(V) \leqq \mathrm{Gini}(X)$ and $W(V) \geqq W(X)$ (See Figure 2-c).

References

Part I.

Nishio, M. (1978). *Theory of Probability* (in Japanese), Jikkyo shuppan, Tokyo.

Moriguti, S. (1995). *Probability Representing Function* (in Japanese), Univ. of Tokyo Press.

Mathematical Society of Japan (eds.) (1985). *Encyclopaedia of Mathematics* (3rd ed.) (in Japanese), Iwanami Shoten Publisher, Tokyo.

Kendall, M.G. and Stuart, A. (1952). *The Advanced Theory of Statistics*, Vol. 1. (3rd ed.), Charles Griffin, London.

Part II.

Evans, M., Hastings, N. and Peacock, B. (1993). *Statistical Distributions* (2nd ed.), Wiley-Interscience Publication.

Atkinson, A.B. (1983). *Social Justice and Public Policy*, Harvester Wheatsheaf.

Chapter 9

Note on Lognormal and Multivariate Normal Distributions

1. Lognormal Distribution

Definition 1.1. A random variable X is called to follow a normal (Gaussian) distribution with parameters $m \in \mathbb{R}$ and $\sigma > 0$ if it has the probability density function:

$$\phi(x; m, \sigma) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}.$$

We denote this by $X \sim N(m, \sigma^2)$.

Proposition 1.2. *The normal distribution $X \sim N(m, \sigma^2)$ has the following properties:*

(i) Mean $E[X] = \text{Mode} = \text{Median} = m$.

(ii) Variance $V[X] = \sigma^2$.

(iii) Moment generating function $E[\exp tX] = \exp\left(mt + \frac{1}{2}\sigma^2 t^2\right)$.

(iv) $X \sim N(m, \sigma^2) \iff (X - m)/\sigma \sim N(0, 1)$.

(v) If $X_i \sim N(m_i, \sigma_i^2)(i = 1, 2, ..., n)$ are independent then the sum (or convolution) $Y = \sum c_i X_i$ also follows a normal distribution with mean $\sum c_i m_i$ and variance $\sum c_i \sigma_i^2$.

Definition 1.3. A random variable X is called to follow a lognormal distribution with parameters $m > 0$ and $\sigma > 0$ if $Y = \log X$ follows the normal distribution $N(m, \sigma^2)$. We denote this by $X \sim LN(m, \sigma^2)$, thus

$$X \sim LN(m, \sigma^2) \iff \log X \sim N(m, \sigma^2) \iff (\log X - m)/\sigma \sim N(0, 1).$$

Proposition 1.4. *Let Y be a random variable and let f_Y, F_Y be its p.d.f. and c.d.f., respectively. If we define a random variable X by $X = p(Y)$ (or $Y = q(X)$), then the p.d.f. and c.d.f. of X, denoted by f_X and F_X, respectively, are given as follows:*

(i) $f_X(x) = f_Y(q(x))q'(x)$.

(ii) $F_X(x) = F_Y(q(x))$.

(iii) $F_X^{-1}(t) = p(F_Y^{-1}(t))$.

[Hint of proof: $f_X(x)dx = f_Y(y)dy$]

Applying this for $Y \sim N(m, \sigma^2)$ with $X = \exp Y,\ Y = \log X$, we have

Theorem 1.5. *Let $\phi(x)$, $\Phi(x)$ be the p.d.f. and c.d.f. of the standard normal distribution $N(0,1)$. A lognormal distribution $X \sim LN(m, \sigma^2)$ has the following properties.*

(i) Probability density function:

$$\lambda(x; m, \sigma) = \phi\left(\frac{\log x - m}{\sigma}\right) \cdot \frac{1}{\sigma x} = \frac{1}{\sqrt{2\pi}\sigma x} e^{-\frac{1}{2}\left(\frac{\log x - m}{\sigma}\right)^2}.$$

(ii) Cumulative distribution function:

$$\Lambda(x; m, \sigma) = \Phi\left(\frac{\log x - m}{\sigma}\right).$$

(iii) Inverse cumulative distribution function (or probability representing function):

$$\Lambda^{-1}(t; m, \sigma) = \exp(m + \sigma\Phi^{-1}(t)).$$

(iv) Mean $E[X] = \exp(m + \frac{1}{2}\sigma^2)$.

(v) Variance $V[X] = \exp(2m + \sigma^2)(\exp(\sigma^2) - 1)$.

(vi) Median $= \exp(m)$.

(vii) Mode $= \exp(m - \sigma^2)$.

Proof. (i)-(iii) are direct consequences of Prop. 1.2. To show (iv) and(v), we note that, for $Y = \log X, E[X^n] = E[\exp(nY)] = \exp(mn + \frac{1}{2}\sigma^2 n^2)$. (from Prop 1.1. (iii)). Therefore,

$$\begin{aligned}
E[X] &= \exp\left(m + \frac{1}{2}\sigma^2\right). \\
V[X] &= E[X^2] - E[X]^2 \\
&= \exp(2m + 2\sigma^2) - \exp(2m + \sigma^2) = \exp(2m + \sigma^2)(\exp(\sigma^2) - 1).
\end{aligned}$$

(vi) By definition of the median and from (iii), Median $= \Lambda^{-1}(1/2; m, \sigma) = \exp(m + \sigma\Phi^{-1}(1/2)) = \exp(m)$ (since $\Phi^{-1}(1/2) = 0$).

(v) The mode z is defined as the value such that $\lambda'(z; m, \sigma) = 0$. From (i) this is equivalent to

$$\phi'\left(\frac{\log z - m}{\sigma}\right) - \sigma\phi\left(\frac{\log z - m}{\sigma}\right) = 0.$$

[Note that $\phi'(x) = -x\phi(x)$]

$$\iff -\left(\frac{\log z - m}{\sigma} + \sigma\right) \cdot \phi\left(\frac{\log z - m}{\sigma}\right) = 0 \iff \log z = m - \sigma^2 \iff z = \exp(m - \sigma^2).$$

Remarks 1.6.

(1) From (iv), (vi) and (vii), it follows that: mode $<$ median $<$ mean. This suggests that the graph of p.d.f. of $LN(m, \sigma^2)$ is a curve skewed to the left.

(2) Let $\alpha = E[X]$ and $\beta^2 = V[X]$, then by solving (iv) and (v) with respect to m and σ, we have

$$m = \log\alpha - \frac{1}{2}\log\left(\frac{\beta^2}{\alpha^2} + 1\right) = \log\alpha - \frac{1}{2}\sigma^2,$$

and

$$\sigma^2 = \log\left(\frac{\beta^2}{\alpha^2} + 1\right).$$

(3) For $0 < a < b$, we have

$$\int_a^b x\lambda(x; m, \sigma)dx = E[X]\left[\Phi\left(\frac{\log b - m}{\sigma} - \sigma\right) - \Phi\left(\frac{\log a - m}{\sigma} - \sigma\right)\right].$$

In fact,

$$LHS = \int_a^b \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{\log x - m}{\sigma}\right)^2\right]dx$$

[changing variable by $y = (\log x - m)/\sigma$, then $dy = dx/(\sigma x)$, and put $y_a = (\log a - m)/\sigma$ and $y_b = (\log b - m)/\sigma$]

$$\begin{aligned} &= \int_{y_a}^{y_b} \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{1}{2}y^2\right)\sigma\exp(m + \sigma y)dy \\ &= e^{m+\frac{\sigma^2}{2}} \cdot \int_{y_a}^{y_b} e^{-\frac{1}{2}(y-\sigma)^2}dy. \end{aligned}$$

Example 1.7. Suppose that gross earnings of a certain insured population is distributed by $LN(m, \sigma^2)$. Consider the minimum and maximum contributory wages, denoted by A and B $(0 < A < B)$, respectively. Then,

(i) The percentage of population whose contributory wage equals the minimum wage is given by

$$\Lambda(A; m, \sigma) = \Phi\left(\frac{\log A - m}{\sigma}\right).$$

(ii) Similarly, the percentage of population whose contributory wage equals the maximum wage is given by

$$1 - \Lambda(B; m, \sigma) = 1 - \Phi\left(\frac{\log B - m}{\sigma}\right).$$

(iii) The average contributory earnings subject to both minimum and maximum wages are given by

$$\begin{aligned}&\int_A^B x\lambda(x; m, \sigma)dx + A \cdot \Lambda(A; m, \sigma) + B \cdot (1 - \Lambda(B; m, \sigma))\\ =&e^{m+\frac{\sigma^2}{2}}\left[\Phi\left(\frac{\log B - m}{\sigma} - \sigma\right) - \Phi\left(\frac{\log A - m}{\sigma} - \sigma\right)\right] + A\Phi\left(\frac{\log A - m}{\sigma}\right) + B\left(1 - \Phi\left(\frac{\log B - m}{\sigma}\right)\right).\end{aligned}$$

Note that the values of the above functions can be calculated numerically. For example, in Excel, one can find $\Phi(t) = \text{NORMSDIST}(t)$ and $\Lambda(t; m, \sigma) = \text{LOGNORMDIST}(t, m, \sigma)$.

Theorem 1.8. *For $X \sim LN(m, \sigma^2)$,*

(i) *The Lorenz curve is given by $x = \Phi(t), y = \Phi(t - \sigma)$ $(-\infty \leqq t \leqq \infty)$.*

(ii) *The Gini coefficient is $G = 2\Phi\left(\frac{\sigma}{\sqrt{2}}\right) - 1 = \int_{-\sigma/\sqrt{2}}^{\sigma/\sqrt{2}} \phi(x)dx.$*

Proof. (i) From Theorem 1.5 (iii), if we denote $\alpha = E[X] = \exp(m + \frac{1}{2}\sigma^2)$, the Lorenz curve is written

$$y = \frac{1}{\alpha}\int_0^x \exp(m + \sigma\Phi^{-1}(u))du.$$

Hence,

$$\frac{dy}{dx} = \frac{1}{\alpha}\exp(m + \sigma\Phi^{-1}(x)).$$

Put $t = \Phi^{-1}(x)$ (i.e. $x = \Phi(t)$), then we have

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{\alpha}\exp(m + \sigma t) \cdot \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}t^2\right) = \frac{1}{\alpha} \cdot \frac{1}{\sqrt{2\pi}}\exp\left(m + \frac{1}{2}\sigma^2 - \frac{1}{2}(t - \sigma)^2\right)\\ &= \frac{1}{\alpha}\exp\left(m + \frac{1}{2}\sigma^2\right) \cdot \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(t - \sigma)^2\right) = \phi(t - \sigma).\end{aligned}$$

Hence, $y = \Phi(t - \sigma)$.

(ii) We have

$$\int_0^1 ydx = \int_{-\infty}^{\infty} \Phi(t - \sigma)\phi(t)dt = \int_{-\infty}^{\infty} \Phi(t - \sigma)\phi(-t)dt = (\Phi * \phi)(-\sigma) = \Theta(-\sigma).$$

Noting that Θ is the c.d.f. of $N(0,1)+N(0,1)=N(0,\sqrt{2})$ (by Prop. 1.1. (v)), we have

$$\Theta(-\sigma)=\Phi(-\sigma;0,\sqrt{2})=1-\Phi(\sigma;0,\sqrt{2})=1-\Phi\left(\frac{\sigma}{\sqrt{2}};0,1\right)=1-\Phi\left(\frac{\sigma}{\sqrt{2}}\right).$$

Hence, the Gini coefficient is calculated as:

$$G=1-2\int_0^1 ydx=1-2\left(1-\Phi\left(\frac{\sigma}{\sqrt{2}}\right)\right)=2\Phi\left(\frac{\sigma}{\sqrt{2}}\right)-1=\int_{-\sigma/\sqrt{2}}^{\sigma/\sqrt{2}}\phi(x)dx.$$

(Q.E.D.)

Table 1.9. Gini coefficients of the lognormal distribution for selected σ

σ	0	$\frac{1}{2}\sqrt{2}$	$\sqrt{2}$	$2\sqrt{2}$	$3\sqrt{2}$	∞
G	0	0.383	0.682	0.954	0.997	1

2. Bivariate Normal Distribution

Definition 2.1. (Multivariate normal distribution)

The probability density function of the multivariate normal distribution is defined by

$$f(x)=(\sqrt{2\pi})^{-n}(\sqrt{\det A})^{-1}\exp\left[-\frac{1}{2}(x-m)^tA^{-1}(x-m)\right].$$

Here, $x=(x_1,......,x_n)^t$ is a vector of n random variables; A is an $n\times n$ positive definite matrix, called covariance matrix, with elements $A_{ij}=\mathrm{cov}(x_i,x_j)=\sigma_{ij}$; and, $m=(m_1,.....,m_n)^t$ is an n-dimensional mean vector

Proposition 2.2. (Formula for $n=2$).

The probability function of bivariate normal distribution, denoted by $f(x_1,x_2)$, is given by:

$$f(x_1,x_2)=\frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\cdot\exp\left(-\frac{1}{2(1-\rho^2)}Q\right),$$

where

$$Q=\left(\frac{x_1-m_1}{\sigma_1}\right)^2-2\rho\left(\frac{x_1-m_1}{\sigma_1}\right)\left(\frac{x_2-m_2}{\sigma_2}\right)+\left(\frac{x_2-m_2}{\sigma_2}\right)^2,$$

and ρ is the correlation coefficient between x_1 and x_2: $\rho=\dfrac{\sigma_{12}}{\sigma_1\sigma_2}$.

Proof. By definition, matrix A is given

$$A=\begin{pmatrix}\mathrm{cov}(x_1,x_1) & \mathrm{cov}(x_1,x_2)\\ \mathrm{cov}(x_2,x_1) & \mathrm{cov}(x_2,x_2)\end{pmatrix}=\begin{pmatrix}\sigma_1^2 & \sigma_{12}\\ \sigma_{21} & \sigma_2^2\end{pmatrix}.$$

Thus,

$$\det A=\sigma_1^2\sigma_2^2-\sigma_{12}\sigma_{21}=\sigma_1^2\sigma_2^2(1-\rho^2).$$

The inverse matrix of A is calculated as

$$A^{-1} = \frac{1}{\sigma_1^2\sigma_2^2(1-\rho^2)}\begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{21} & \sigma_1^2 \end{pmatrix} = \frac{1}{1-\rho^2}\begin{pmatrix} \frac{1}{\sigma_1^2} & -\frac{\rho}{\sigma_1\sigma_2} \\ -\frac{\rho}{\sigma_1\sigma_2} & \frac{1}{\sigma_2^2} \end{pmatrix}.$$

Thus,

$$\begin{aligned}
&\frac{1}{2}(x-m)^t A^{-1}(x-m) \\
&= -\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-m_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-m_1}{\sigma_1}\right)\left(\frac{x_2-m_2}{\sigma_2}\right) + \left(\frac{x_2-m_2}{\sigma_2}\right)^2\right].
\end{aligned}$$

Therefore, we obtain the required formula. (Q.E.D.)

Corollary 2.3.

(i) *The marginal distribution of* x_j *is the univariate normal distribution with mean* m_j *and variance* σ_j^2 $(j = 1, 2)$.

(ii) *The conditional probability density function of* X_1 *when* $X_2 = x_2$ *is given by*

$$f(x_1|X_2 = x_2) = \frac{1}{\sqrt{2\pi\sigma_1^2(1-\rho^2)}}\exp\left[-\frac{1}{2\sigma_1^2(1-\rho^2)}\left((x_1-m_1) - \rho\frac{\sigma_1}{\sigma_2}(x_2-m_2)\right)^2\right].$$

Thus, the conditional distribution of x_1 *given* x_2 *is the univariate normal distribution whose mean and variance are given by*

$$E[X_1|X_2 = x_2] = m_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2 - m_2) \qquad \text{and} \qquad V[X_1|X_2 = x_2] = \sigma_1^2(1-\rho^2).$$

Proof. (i) By rearranging the exponent of $f(x_1, x_2)$, we have

$$-\frac{1}{2(1-\rho^2)}Q = -\frac{1}{2\sigma_1^2(1-\rho^2)}\left[(x_1-m_1) - \rho\frac{\sigma_1}{\sigma_2}(x_2-m_2)\right]^2 - \frac{1}{2}\left(\frac{x_2-m_2}{\sigma_2}\right)^2.$$

By using the formula $\int_{-\infty}^{\infty}\exp\left[-\left(\frac{x}{a}\right)^2\right]dx = a\sqrt{\pi}$ (for $a > 0$), it can be shown that the marginal probability density function $f(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\,dx_1$ is the univariate normal distribution with mean m_2 and variance σ_2^2.

(ii) From the result of (i), one can calculate the conditional probability density function by $f(x_1|X_2 = x_2) = f(x_1, x_2)/f(x_2)$. (Q.E.D.)

3. Application

Suppose that we estimate the amount of pensions for newly retired workers. Usually, the pension formula is a function of the length of contribution period and the salary (referring, for instance, to the last drawn salary, the average of the salaries of several years before retirement, or the average of the re-evaluated salaries of the whole working period). Therefore, we should consider the two-dimensional distribution with respect to the salary and the length of work for newly retired workers.

In order to determine a model distribution, we make two assumptions. First, the contribution period is assumed to follow the normal distribution. If the probability of paying contribution for a year is constant ($= p =$ density times collection rate) and this process is independent with respect to time n, then the number of contribution years is given by the Bernoulli process $B(n,p)$ if n is sufficiently large, this can be approximated by a normal distribution. Second, the reference salary is assumed to follow a lognormal distribution. This assumption is supported by income statistics in some countries.

In the following, we find the explicit formula of the above two-dimensional distribution - normal with respect to one variable and lognormal with respect to the other - by using the results obtained in the preceding sections.

Proposition 3.1. *Let $F(x,y)$ be a probability density function of a two dimensional random variable (x,y). Suppose a one-to-one smooth mapping Φ such that $(X,Y) = \Phi(x,y) = (u(x,y), v(x,y)) \iff (x,y) = \Phi^{-1}(X,Y) = (\alpha(X,Y), \beta(X,Y))$ Then the probability density function of (X,Y), denoted by $G(X,Y)$, is given by*

$$G(X,Y) = F(\alpha(X,Y), \beta(X,Y)) \left| \frac{\partial(\alpha,\beta)}{\partial(X,Y)} \right|.$$

Proposition 3.2. *Let $f(x,y)$ be the probability density function of a bivariate normal distribution. If we take the logarithmic scale of Y, then the probability density function is transformed by*

$$g(X,Y) = f(X, \log Y) \cdot \frac{1}{Y} = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot \frac{1}{Y} \cdot \exp\left(-\frac{1}{2(1-\rho^2)} R\right),$$

where

$$R = \left(\frac{X-m_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{X-m_1}{\sigma_1}\right)\left(\frac{\log Y - m_2}{\sigma_2}\right) + \left(\frac{\log Y - m_2}{\sigma_2}\right)^2.$$

Proof. We use the same notation as Proposition 2.2. The changing of the scale of y to the logarithmic one is achieved by the following transformation:

$$\begin{cases} X = u(x,y) = x & ; \quad x = \alpha(X,Y) = X \\ Y = v(x,y) = \exp y & ; \quad y = \beta(X,Y) = \log Y. \end{cases}$$

Hence,

$$\begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & \frac{1}{Y} \end{vmatrix} = \frac{1}{Y}.$$

Direct application of Proposition 3.1. will yield the required results. (Q.E.D.)

Remark. For numerical calculations, an Excel VBA programme using the two-dimensional Simpson's formula, is shown in the Appendix.

Proposition 3.3. *In the above notation, the correlation coefficient between X and Y, denoted by λ, is given by*

$$\lambda = \rho \frac{\alpha}{\beta} \sqrt{\log\left(\frac{\beta^2}{\alpha^2} + 1\right)} = \rho \cdot \sqrt{\frac{\sigma_2^2}{\exp(\sigma_2^2 - 1)}}.$$

Here, $\alpha = E[Y]$ and $\beta^2 = V[Y]$. Note that from Remark 1.6 (2), $m_2 = \log\alpha - \frac{1}{2}\log\left(\frac{\beta^2}{\alpha^2}+1\right) = \log\alpha - \frac{1}{2}\sigma_2^2$ and $\sigma_2^2 = \log\left(\frac{\beta^2}{\alpha^2}+1\right)$.

Proof. To see the effect on the correlation coefficient associated with the change of variables in the bivariate normal distribution, we use the following result (See Appendix 2):

[Stein's lemma] *Suppose that two random variables x and y follow a bivariate normal distribution. Let $\varphi(y)$ be a differentiable function and for any $a > 0$, $\varphi(y) = o\left(\exp(ay^2)\right)$ (as $|y| \to \infty$). Then*

$$\operatorname{cov}(x, \varphi(y)) = E[\varphi'(y)] \cdot \operatorname{cov}(x, y).$$

We shall apply this result to $X = x$ and $Y = \varphi(y) = \exp y$. Noting that $\varphi'(y) = \exp y = \varphi(y) = Y$, the correlation coefficient λ is given by

$$\lambda = \frac{\operatorname{cov}(X,Y)}{\sqrt{V[X]}\sqrt{V[Y]}} = \frac{E[Y]\operatorname{cov}(x,y)}{\sqrt{V[x]}\sqrt{V[Y]}} = \frac{\alpha\cdot\rho\sigma_1\sigma_2}{\sigma_1\beta} = \rho\frac{\alpha}{\beta}\sqrt{\log\left(\frac{\beta^2}{\alpha^2}+1\right)} = \rho\cdot\sqrt{\frac{\sigma_2^2}{\exp(\sigma_2^2 - 1)}}.$$

(Q.E.D.)

Remark. Since $\exp(x) \geqq 1 + x$ (for $x \geqq 0$), it follows that $|\lambda| \leqq |\rho| \leqq 1$. Further, it is seen that ρ and λ take the same sign.

Example 3.4. The following example attempts to illustrate the effect of correlation between the salary level and the length of services on the average pension amount and the percentage of minimum pensions.

Suppose a group of newly retired workers. Concerning their pension related data, assume

- The length of contribution years follows the normal distribution with its average equal to 25 years and its standard deviation equal to 5 years.
- The reference salary for the pension award follows the lognormal distribution with its average \$1000 and its standard deviation \$500.

Concerning the pension formula, assume

- Qualifying period for pensions is 15 years.
- The pension amount is equal to the sum of 2% of the reference salary for each year of contribution.
- Minimum pension is equal to $250.

The Table below shows the results under different assumptions on the correlation coefficient.

Correlation ρ (λ)	Average pension	Percentage of minimum pensioners	Percentage of disqualified workers for pension
0.95 (0.90)	$546 [109]	8.85% [-2.06 points]	2.28%
0.50 (0.47)	$526 [105]	10.05% [-0.85 points]	2.28%
0.00 (0.00)	$503 [100]	10.91% [± 0]	2.28%
-0.50 (-0.47)	$478 [95]	11.16% [+0.25 points]	2.28%
-0.95 (-0.90)	$456 [91]	11.17% [+0.26 points]	2.28%

The following observations can be made:

- The average pension is positively correlated with ρ (or λ), which is a consequence of the fact that the pension formula is bilinear with respect to the reference salary and the contribution years. In the case of the above example, changing ρ (or λ) may deviate ± 9 percent of the average pension from the zero correlation.
- For the same reason as above, the percentage of minimum pensioners shows negative correlation with respect to ρ (or λ). The impact depends also on the amount of the minimum pension. (Note that the impact is not linear in this case).
- The number of workers who fail to meet the qualifying condition for pensions is not affected by the correlation. This is obvious as the qualifying condition is concerned only with the length of contribution years, whose (marginal) distribution should be identical for all cases.

Concluding remark. In actuarial projections, there is a growing need to take into account distributional effects in the estimation of benefit costs. This paper has presented one of the simplest and most tractable models.

References

Kendall, M.G. and Stuart, A. (1952). *The Advanced Theory of Statistics*, Vol. 1. (3rd ed.), Charles Griffin, London.

Aitchinson, J. and Brown, J.A.C. (1957). *The lognormal distribution*, Cambridge Univ. Press.

Evans, M., Hastings, N. and Peacock, B. (1993). *Statistical Distributions* (2nd ed.), Wiley-Interscience Publication.

Limpert, E., Stahl, W.A. and Abbt, M. (2001). "Lognormal distributions across the sciences: keys and clues", Bioscience, 51(5):341-352.
(available at `http://linkage.rockefeller.edu/wli/zipf/limpert01.pdf`)

Appendix 1 : An Excel VBA programme to calculate the two dimensional normal-lognormal distributions

```
'------------------------------------
'Normal-Lognormal Distribution
'2-dim compound Simpson rule
'------------------------------------

Option Base 0
DefDbl A-H, K-M, O-Z
DefInt I-J, N

Global mx, sx, my1, sy1, cov

Sub Integration()

x0 = Cells(2, 2).Value      'normal distribution'
lx = Cells(3, 2).Value
numx = Cells(4, 2).Value
mx = Cells(5, 2).Value
sx = Cells(6, 2).Value

y0 = Cells(2, 3).Value      'lognormal distribution'
ly = Cells(3, 3).Value
numy = Cells(4, 3).Value
my = Cells(5, 3).Value
sy = Cells(6, 3).Value

my1 = Log(my) - 0.5 * Log(((sy ^ 2) / (my ^ 2)) + 1)    'av. & st. dev. for using normal dist.'
sy1 = (Log(((sy ^ 2) / (my ^ 2)) + 1)) ^ 0.5            'formula for lognormal'

cov = Cells(1, 5).Value     'covariance (rho)'
row1 = Cells(2, 5).Value    'first row and column of generated table'
col1 = Cells(3, 5).Value

Cells(row1, col1).FormulaR1C1 = "Cov   " & cov

lamda = cov * (sy1 ^ 2 / (Exp(sy1 ^ 2) - 1)) ^ 0.5

Cells(7, 5) = lamda

nx = 50         'steps for integration'
ny = 50
nx2 = 2 * nx
ny2 = 2 * ny

For ix = 1 To numx

    If ix = 1 Then
        a = mx - 10 * sx   'first a ~ left tail of function'
    Else: a = x0 + (ix - 2) * lx
    End If
    If ix = numx Then
        b = mx + 10 * sx      'last b ~ right tail of function'
    Else: b = x0 + (ix - 1) * lx
    End If

    'table formatting:'
    If ix = 1 Then
        Cells(row1, (col1 + ix)).FormulaR1C1 = "    -  " & b
    Else
        If ix = numx Then
            Cells(row1, (col1 + ix)).FormulaR1C1 = a & "  -  "
        Else: Cells(row1, (col1 + ix)).FormulaR1C1 = a & "  -  " & b
        End If
    End If

    hx = (b - a) / nx2

    For iy = 1 To numy

        If iy = 1 Then
            c = 0.0000000000001              'first c ~ 0'
```

```
        Else: c = y0 + (iy - 2) * ly
        End If
        If iy = numy Then
            d = my + 20 * sy           'last d ~ tail of function'
        Else: d = y0 + (iy - 1) * ly
        End If

        'table formatting:'
        If iy = 1 Then
            Cells((row1 + iy), col1).FormulaR1C1 = "0  -  " & Int(d)
        Else
            If iy = numy Then
                Cells((row1 + iy), col1).FormulaR1C1 = Int(c) & "  -  "
            Else: Cells((row1 + iy), col1).FormulaR1C1 = Int(c) & "  -  " & Int(d)
            End If
        End If

'approximation of integral:'
        hy = (d - c) / ny2
        AJ1 = 0: AJ2 = 0: AJ3 = 0

        For i = 0 To nx2
            x1 = a + i * hx
            K1 = Lognorm(x1, c) + Lognorm(x1, d)
            K2 = 0: K3 = 0

            For j = 1 To (ny2 - 1)
                y1 = c + j * hy
                QQ = Lognorm(x1, y1)
                If j / 2 - Fix(j / 2) = 0 Then
                    K2 = K2 + QQ
                Else: K3 = K3 + QQ
                End If
            Next j

            ANS1 = (K1 + 2 * K2 + 4 * K3) * hy / 3
            If (i = 0) Or (i = nx2) Then
                AJ1 = AJ1 + ANS1
            Else
                If i / 2 - Fix(i / 2) = 0 Then
                    AJ2 = AJ2 + ANS1
                Else: AJ3 = AJ3 + ANS1
                End If
            End If
        Next i

        ANS2 = (AJ1 + 2 * AJ2 + 4 * AJ3) * hx / 3
        Cells((row1 + iy), (col1 + ix)).Value = ANS2
    Next iy
Next ix

End Sub

Function Lognorm(x#, y#)

p# = 1 / ((2 * 3.14159265358979) * ((sx ^ 2 * sy1 ^ 2 * (1 - cov ^ 2))) ^ (0.5))
q# = (x# - mx) ^ 2 * sy1 ^ 2 - 2 * cov * (x# - mx) * (Log(y#) - my1) * sx * sy1
    + (Log(y#) - my1) ^ 2 * sx ^ 2
r# = -2 * sx ^ 2 * sy1 ^ 2 * (1 - cov ^ 2)

Lognorm = p# * Exp(q# / r#) * (1 / y#)

End Function
```

Appendix 2 : Proof of Stein's lemma

Proposition. *Suppose that two random variables x and y follow a bivariate normal distribution. Let $\varphi(x)$ be a differentiable function which satisfies the condition: $\lim_{|x|\to\infty} \varphi(x) \cdot \exp(-ax^2) = 0$ (for any $a > 0$). Then we have*

$$\mathrm{cov}(\varphi(x), y) = E[\varphi'(x)] \cdot \mathrm{cov}(x, y).$$

Proof. Let m_X, σ_X^2, f_X denote mean, variance and p.d.f. of X, respectively. (Similarly for Y)

Recall first,

$$\mathrm{cov}(x, y) = \mathrm{cov}(x, E[y|x]),$$

which is one variant of the law of iterated expectations.

Second, from Corollary 2.3.(ii),

$$E[y|x] = m_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - m_X).$$

Third, noting that $f_X(x) = \sigma_X^{-1} \cdot \phi((x - m_X)/\sigma_X)$ and $\phi'(x) = -x\phi(x)$, we have:

$$(x - m_X)f_X(x) = -\sigma_X^2 f_X'(x).$$

Hence,

$$\begin{aligned}
\mathrm{cov}(\varphi(x), y) &= \mathrm{cov}(\varphi(x), E[y|x]) \\
&= \mathrm{cov}(\varphi(x), m_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - m_X)) = \rho\frac{\sigma_Y}{\sigma_X}E[\varphi(x)(x - m_X)] \\
&= \rho\frac{\sigma_Y}{\sigma_X}\int_{-\infty}^{\infty}\varphi(x)(x - m_X)f_X(x)dx = -\rho\sigma_X\sigma_Y\int_{-\infty}^{\infty}\varphi(x)f_X'(x)dx \\
&= -\mathrm{cov}(x, y)\left[\varphi(x)f_X(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\varphi'(x)f_X(x)dx\right] = \mathrm{cov}(x, y)E[\varphi'(x)].
\end{aligned}$$

(Q.E.D.)

Note: Corollary 2.3. (ii) implies that if we regress y on x, then we have

$$y = m_Y + \rho\frac{\sigma_Y}{\sigma_X}(x - m_X) + \varepsilon,$$

where $\varepsilon|x \sim N(0, \sigma_Y^2(1 - \rho^2))$. Note that ε is independent of x hence $\varepsilon \sim N(0, \sigma_Y^2(1 - \rho^2))$ and $E[\varepsilon|x] = E[\varepsilon] = 0$.

Chapter 10

On Some Issues in Actuarial Mathematics

This paper consists of three parts. Part I aims at clarifying some mathematical formulae of the infinitesimal and finite difference calculus related to actuarial mathematics. We will show that the Taylor series expansion, Newton's interpolation formula, the Euler-Maclaurin summation formula and the Woolhouse summation formula can be understood in a unified manner in terms of a formal relation: $e^{\partial} = 1 + \Delta$, where $\partial = d/dx$ is the differential operator and Δ, the forward finite difference operator. In Part II, we present basic results in the numerical analysis - polynomial interpolation, numerical differentiation, and numerical integration - in the context of an application of finite difference method to differential and integral calculus. In the Conclusion, we discuss some examples in actuarial mathematics. Throughout this paper, all functions are defined on $\mathbb{R}$ unless otherwise specified.

Part I. Topics in Infinitesimal and Finite Difference Calculus

1. Taylor Series Expansion and Analytic Functions

1.1 Introduction

Suppose a function $f(x)$ given by an absolutely convergent power series with centre at $x = a$:

$$f(x) = A_0 + A_1(x-a) + A_2(x-a)^2 + \cdots + A_n(x-a)^n + \cdots .$$

The coefficients A_n $(n = 0, 1, 2, \ldots)$ are determined in the following manner:

First, we note that $f(a) = A_0$.

Differentiating the above equation (the term-wise differentiation is justified for an absolutely convergent power series), we have

$$f'(x) = A_1 + 2A_2(x-a) + 3A_3(x-a)^2 + \cdots + nA_n(x-a)^{n-1} + \cdots .$$

Putting $x = a$, we have $f'(a) = A_1$.

By applying this operation successively, we have: $f^{(n)}(a) = n!A_n$. Hence, we get:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

The above type of power series is called the Taylor series.

Noting that $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$, this fact can be summarised in the following symbolic notation :

$$f(x+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} \cdot h^n = e^{h\partial} \cdot f(x).$$

In the above expression, we understand that $e^{h\partial}$ has a similar expansion as the exponential function and $\partial^n f(x)$ means the n-th derivative of $f(x)$.

In particular, in the case of $h = 1$, we have $f(x+1) = e^{\partial} f(x) = (1+\Delta) \cdot f(x)$. This can be symbolically denoted by $e^{\partial} = 1 + \Delta$.

1.2 Taylor's formula with remainder

We give more precise formulation of the Taylor series expansion. Although this formula is generally well-known, we give a proof as a prototype for the subsequent discussions.

Theorem I.1.1. *If a function $f(x)$ is differentiable for n times and $f^{(n)}(x)$ is continuous on an open interval containing $[a, b]$, then*

$$f(b) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(b-a)^k + R_n(b),$$

where the reminder is given by

$$R_n(b) = \frac{1}{(n-1)!} \int_a^b (b-t)^{n-1} f^{(n)}(t) dt.$$

Proof. We start from

$$f(b) - f(a) = \int_a^b f'(t) dt.$$

Applying integration by parts with $v(t) = f(t)$ and $u(t) = -(b-t)$, we get

$$f(b) - f(a) = (b-a)f'(t) + \int_a^b (b-t) f''(t) dt.$$

Applying again integration by parts for the above defined integral with $v(t) = f'(t)$ and $u(t) = -\frac{1}{2}(b-t)^2$, we have

$$f(b) - f(a) = (b-a)f'(a) + \frac{1}{2}(b-a)^2 f''(a) + \int_a^b \frac{1}{2}(b-t)^2 f'''(t)dt.$$

Continuing this procedure, we arrive at the desired result. (Q.E.D.)

1.3 Relation between infinitely differentiable functions and real analytic functions

A function $f(x)$ is called (real) analytic on (a,b) if for any point $x \in (a,b)$, f is expanded by an absolutely convergent power series in a neighbourhood of x. (In fact, as seen in the introduction, this series is identical to the Taylor series.) A function $f(x)$ is called infinitely differentiable on (a,b) if f is continuously differentiable for any times on (a,b).

Theorem I.1.2. *Let f be an infinitely differentiable function on (a,b). Then, the following condition [#] is a sufficient and necessary condition for f to be analytic on (a,b) :*

[#] *For any $\xi \in (a,b)$, there exists a closed neighbourhood of $\xi : N(\xi) = \{x; |x-\xi| \leqq \varepsilon\}$, and there exist $M > 0$ and $r > 0$ such that $|f^{(n)}(x)| \leqq Mn!r^{-n}$ (for all $x \in N(\xi)$ and for all $n \geqq 0$).*

Proof. If f is analytic, then f has a power series expansion in a neighbourhood of $\xi : f(x) = \sum a_k(x-\xi)^k$. Choose a point $x_0 \neq \xi$ in the neighbourhood of ξ and put $d = |x_0 - \xi|$ and $\mu = \sum |a_k| \cdot |x_0 - \xi|^k$ (note that $|f(x_0)| \leqq \mu < \infty$). Fix any q such that $0 < q < 1$ (put, for example, $q = 1/2$), and put $N(\xi) = \{x; |x-\xi| \leqq qd\}$.

For any $x \in N(\xi)$,

$$\begin{aligned}
|f^{(n)}(x)| &= \left| \left(\frac{d}{dx}\right)^n \sum_{k=0}^{\infty} a_k(x-\xi)^k \right| = \left| \sum_{k=n}^{\infty} a_k \frac{k!}{(k-n)!}(x-\xi)^{k-n} \right| \\
&\leqq \left| \sum_{k=n}^{\infty} a_k \frac{k!}{(k-n)!} q^{k-n}(x_0-\xi)^{k-n} \right| \leqq \left| \sum_{k=0}^{\infty} a_k(x_0-\xi)^{k-n} \cdot \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} q^{k-n} \right| \\
&\leqq \mu d^{-n} \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} q^{k-n}.
\end{aligned}$$

Further, for $|q| < 1$ and $n \geqq 0$,

$$\sum_{k=n}^{\infty} \frac{k!}{(k-n)!} q^{k-n} = \sum_{k=n}^{\infty} \left(\frac{d}{dq}\right)^n q^k = \left(\frac{d}{dq}\right)^n \sum_{k=0}^{\infty} q^k = \left(\frac{d}{dq}\right)^n \frac{1}{1-q} = \frac{n!}{(1-q)^{n+1}}.$$

By combining the above results, we have shown

$$|f^{(n)}(x)| \leqq \mu d^{-n} \frac{n!}{(1-q)^{n+1}}.$$

Putting $M = \mu/(1-q)$ and $r = d(1-q)$ leads to the condition [#].

Conversely, suppose an infinitely differentiable function f satisfies the condition [#]. For any point $\xi \in (a,b)$, we can find $\varepsilon = \varepsilon(\xi)$ and $r = r(\xi)$ which satisfy the condition [#]. Put $U(\xi) = \{x; |x-\xi| \leqq p\delta\}$, where $\delta = \min(\varepsilon, r) > 0$ and $0 < p < 1$. From the reminder of Taylor's formula of order n at $x \in U(\xi)$, we have

$$\begin{aligned}|R_n(x)| &= \frac{1}{(n-1)!}\left|\int_\xi^x (x-t)^{n-1} f^{(n)}(t)dt\right| \\ &\leqq \frac{Mn!r^{-n}}{(n-1)!}\left|\int_\xi^x (x-t)^{n-1}dt\right| \leqq M\cdot\left|\frac{x-\xi}{r}\right|^n \leqq M\cdot p^n.\end{aligned}$$

Since $0 < p < 1$, the reminder $R_n(x)$ converges uniformly to zero on $U(\xi)$ as n tends to infinity. Therefore, f has the Taylor series expansion on $U(\xi)$.

To show the absolute convergence of this series, denoted by $P(x) = \sum_{k=0}^\infty a_k(x-\xi)^k$, choose a point $x_0 \in U(\xi) \setminus \{\xi\}$. Since $P(x)$ converges at $x = x_0$, there exists $R > 0$ such that $|a_k|\cdot|x_0-\xi|^k \leqq R$ (for all $k \geqq 0$). Hence, on the set $\{x;\ |x-\xi| \leqq s|x_0-\xi|,\ 0 < s < 1\}$, we have $\sum_{k=0}^\infty |a_k|\cdot|x-\xi|^k \leqq R\sum_{k=0}^\infty s^k = R/(1-s) < \infty$. (Q.E.D.)

Remark 1. Suppose f is real analytic on $U(\xi) = \{x \in \mathbb{R}; |x-\xi| \leqq \delta\}$. The above discussion is valid for a complex variable if we interpret $|z| = \sqrt{z\overline{z}}$ ($\overline{z}$ is the complex conjugate of z). Thus, f is expressed as an absolutely convergent power series on $\tilde{U}(\xi) = \{z \in \mathbb{C}; |z-\xi| \leqq \delta\}$ (i.e., f is a complex analytic function). In this way, any real analytic function on a real interval can be extended to a complex analytic function on an open set in $\mathbb{C}$ containing that interval. In other words, a real analytic function can be considered as a complex analytic function whose argument is restricted to the real axis.

Remark 2. For a complex power series $P(z) = \sum_{k=0}^\infty a_k(z-\xi)^k$ ($\xi \in \mathbb{C}$), $\rho = \rho(\xi) = \sup\{|z-\xi|;\ P(z)$ converges absolutely $\}$ is called its radius of convergence. Using knowledge of the theory of complex functions, the first part of the above proof (the necessity of [#]) is simplified as follows.

Suppose f is real analytic, then it can be extended to a complex analytic function (denoted $f(z)$, where z is a complex variable). For any $\xi \in (a,b)$, the radius of convergence of $f(z)$ at $z = \xi$, $\rho = \rho(\xi)$ is strictly positive. Choose $r > 0$ such that $0 < 2r < \rho$ and put $M_0 = \max\{|f(z)|; |z-\xi| = 2r\}$.

By Cauchy's integration formula, for $|z-\xi| \leqq r$ and for $n \geqq 0$, we have

$$|f^{(n)}(z)| = \left|\frac{n!}{2\pi i}\int_{|\zeta-\xi|=2r}\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta\right| \leqq \frac{M_0 n!}{2\pi}\int_{|\zeta-\xi|=2r}\frac{|d\zeta|}{|\zeta-z|^{n+1}} \leqq 2M_0 n! r^{-n}.$$

Here, we have used the fact that for any ζ and z such that $|\zeta-\xi| = 2r$ and $|z-\xi| \leqq r$ respectively, it follows that $|\zeta-z| \geqq r$. Thus, taking $N(\xi) = \{x \in \mathbb{R}; |x-\xi| \leqq r\}$ and $M = 2M_0$ will yield condition [#].

Remark 3. Let f be an infinitely differentiable function on (a,b). Then, the following condition [##] gives a sufficient condition for f to be analytic on (a,b):

[##] There exist $K > 0$ and $h > 0$ such that $|f^{(n)}(x)| \leqq Kh^{-n}n!$ (for all $x \in (a, b)$ and for all $n \geqq 0$).

The above condition can be rephrased as:

[##′] For some $h > 0$, $\|f^{(n)}\|_\infty = O(h^{-n}n!)$ (as $n \to \infty$),

where $\|f^{(n)}\|_\infty = \sup\limits_{x\in(a,b)} |f^{(n)}(x)|$.

Note. The meaning of notation O and o, called Landau's symbols, is as follows:

- $f(x) = O(g(x))$ as $x \to a$. $\iff$ There exist $K > 0$ and a neighbourhood of $a, N(a)$, such that $|f(x)| \leqq Kg(x)$ (for all $x \in N(a)$).
- $f(x) = o(g(x))$ as $x \to a$. $\iff$ For any $\varepsilon > 0$, there exists a neighbourhood of $a, N(a)$, such that $|f(x) \leqq \varepsilon g(x)$ (for all $x \in N(a)$). $\iff$ $\lim_{x\to a} f(x)/g(x) = 0$.

We may omit "as $x \to a$", when it is clear from the context.

In the case of series, $\{f_n\}$, a similar definition applies.

- $f_n = O(g_n)$ as $n \to \infty$. $\iff$ There exist $K > 0$ and $N \in \mathbb{N}$ such that $|f_n| \leqq Kg_n$ (for all $n \geqq N$).
- $f_n = o(g_n)$ as $n \to \infty$. $\iff$ For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|f_n| \leqq \varepsilon g_n$ (for all $n \geqq N$). $\iff$ $\lim_{n\to\infty} f_n/g_n = 0$.

Similarly, we omit "as $n \to \infty$", when it is clear from the context.

2. Calculus of Finite Difference and Newton's Formula

2.1 Elements in calculus of finite differences

In this section, we give a finite difference version of Taylor's theorem for polynomials. First, we summarise here some basic properties of the finite difference operator. In contrast to the continuous case, the discrete case may involve some asymmetry such as shifts in variables or lag in upper or lower bounds.

Recall the definitions:

Shift:
$$Ef(x) = f(x + 1) = (1 + \Delta)f(x).$$

Forward difference:
$$\Delta f(x) = f(x + 1) - f(x) = (E - 1)f(x).$$

Then, the following propositions hold in parallel with the case of infinitesimal analysis:

(i) Higher order differences:

$$\Delta^n f(x) = \Delta(\Delta^{n-1} f(x)) = (E-1)^n f(x) = \sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} f(x+k).$$

(ii) Difference of a product:

$$\Delta[f(x)g(x)] = \Delta f(x)g(x) + f(x+1)\Delta g(x) = \Delta f(x)g(x) + f(x)\Delta g(x) + \Delta f(x)\Delta g(x).$$

(iii) Higher order of differences of a product (Leibniz's rule):

$$\Delta^n f(x)g(x) = \sum_{k=0}^{n} \binom{n}{k} \Delta^{n-k} f(x+k)\Delta^k g(x).$$

(iv) Summation: If $\Delta F(x) = f(x)$ then

$$\sum_{x=1}^{n} f(x) = F(n+1) - F(1) =: \Delta^{-1} f(x)\Big|_{x=1}^{n+1}.$$

(v) Summation by parts:

$$\sum_{x=1}^{n} \Delta f(x)g(x) = (f(n+1)g(n+1) - f(1)g(1)) - \sum_{x=1}^{n} f(x+1)\Delta g(x).$$

2.2 Newton's formula

Theorem I.2.1. *If $p(x)$ is a polynomial of degree n, then*

$$p(x) = \sum_{k=0}^{n} \binom{x-a}{k} \Delta^k p(a).$$

Proof. The proof runs in parallel to that of Theorem 1.1.1.

Before going into the proof, we note the following relation, which can be verified by direct calculation:

$$\Delta_{(i)} \binom{x-i}{k} = -\binom{x-i-1}{k-1},$$

where, $\Delta_{(i)}$ stresses that the forward difference is taken with respect to i.

We start from

$$p(x) - p(a) = \sum_{i=a}^{x-1} \Delta p(i).$$

Applying summation by parts with $g(i) = \Delta p(i)$ and $f(i) = -(x-i)$, we get

$$p(x) - p(a) = (x-a)\Delta p(a) + \sum_{i=a}^{x-1} (x-i-1)\Delta^2 p(i) = \binom{x-a}{1} \Delta p(a) + \sum_{i=a}^{x-1} \binom{x-i-1}{1} \Delta^2 p(i).$$

Applying again summation by parts with $g(i) = \Delta^2 p(i)$ and $f(i) = -\frac{1}{2}(x-i)(x-i-1)$, we get

$$p(x) - p(a) = \binom{x-a}{1} \Delta p(a) + \binom{x-a}{2} \Delta^2 p(a) + \sum_{i=a}^{x-1} \binom{x-i-1}{1} \Delta^3 p(i).$$

Continuing this procedure, we arrive at the desired result. Note that after applying this for n times the reminder term will disappear, as $\Delta^{n+1} p = 0$. (Q.E.D.)

Remark. In symbol notation, Newton's formula is written as:

$$f(x+h) = \sum_{k=0}^{\infty} \binom{h}{k} \Delta^k f(x) = (1+\Delta)^h f(x).$$

In the above expression, we understand that $(1+\Delta)^h$ has a similar expansion as the binomial series and $\Delta^k f(x)$ means the k-th difference of $f(x)$. Since $\Delta^k f(x) = 0$ (for $k > \deg f$), the series terminates within a finite number of terms.

Further, we introduce the $\frac{1}{m}$ -difference operator:

$$\Delta^{(m)} \cdot f(x) = m\left(f\left(x + \frac{1}{m}\right) - f(x)\right) = m((1+\Delta)^{\frac{1}{m}} - 1) \cdot f(x).$$

Note that $\Delta^{(1)} f(x) = \Delta f(x)$ and $\lim_{m \to \infty} \Delta^{(m)} f(x) = f'(x) = \partial f(x)$.

Comparing Theorems 1.1 and 1.2, we have the following formal identity:

$$e^{h\partial} = (1+\Delta)^h = \left(1 + \frac{\Delta^{(m)}}{m}\right)^{mh}.$$

2.3 An algebraic proof of $\partial = \log(1+\Delta)$

Solving formally the equation $e^{\partial} = 1 + \Delta$ with respect to ∂, we have the following symbol formula:

$$\partial = \log(1+\Delta) = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \cdots.$$

Note that the series of operator on the right-hand side of the above equation is not *a priori* defined. However, for polynomials, the following formula holds.

Theorem I.2.2. *If $p(x)$ is a polynomial of degree n, then*

$$\frac{d}{dx} p(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \Delta^k p(x).$$

Note that $\Delta^k p(x) = 0$ (for $k > n$).

Proof. In view of the linearity of the difference operator, it is sufficient to show that the desired relation holds for $p(x) = x^n$. In this case, the right-hand side of the statement can be written as:

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \Delta^k x^n = \sum_{k=1}^{n} \left[\frac{(-1)^{k-1}}{k} \cdot \sum_{i=0}^{k} \left(\binom{k}{i} (-1)^{k-i} \cdot \sum_{j=0}^{n} \binom{n}{j} i^{n-j} x^j \right) \right].$$

Let a_j be the coefficient of x^j in the above, then

$$a_j = \binom{n}{j} \sum_{k=1}^{n} \left[\frac{(-1)^{k-1}}{k} \cdot \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} i^{n-j} \right] = \binom{n}{j} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \cdot \Delta^k (x^{n-j}) \Big|_{x=0}.$$

From this, we see that:

(1) $a_n = 0$ (because $\Delta^k 1 = 0$ for every $k \geqq 1$); and,

(2) $a_{n-1} = n$ (because $\Delta x = 1$ and $\Delta^k x = 0$ for every $k \geqq 2$).

As $(x^n)' = nx^{n-1}$, in order to complete the proof, it is sufficient to show $a_j = 0$ (for $j = 0, 1, \ldots, n-2$).

By further calculation, for $j = 0, 1, \ldots, n-2$,

$$a_j = \binom{n}{j} \sum_{k=1}^{n} \left[(-1)^k \sum_{i=0}^{k} \binom{k-1}{i-1} (-1)^{k-i-1} i^{n-j-1} \right] = \binom{n}{j} \sum_{k=0}^{n-1} (-1)^{k+1} \Delta^k (x^{n-j-1}) \Big|_{x=1}.$$

To show that the RHS is equal to 0, it is sufficient to prove the following lemma:

Lemma. *For* $h \geqq 1$,

$$\sum_{k=0}^{h} (-1)^k \cdot \Delta^k (x^h) \Big|_{x=1} = 0.$$

Proof of Lemma. We show by induction of h. Firstly, the above relation clearly holds for $h = 1$. In fact, $(-1)^0 \cdot 1 + (-1)^1 \cdot 1 = 0$.

Next, assume that this statement holds for $h = 1, 2, \ldots, r$. Then,

$$\begin{aligned} \sum_{k=0}^{r+1} (-1)^k \cdot \Delta^k (x^{r+1}) \Big|_{x=1} &= 1 + \sum_{k=0}^{r} (-1)^{k+1} \cdot \Delta^k (\Delta x^{r+1}) \Big|_{x=1} \\ &= 1 - \sum_{k=0}^{r} (-1)^k \cdot \Delta^k \left(1 + \sum_{j=1}^{r} \binom{r+1}{j} x^j \right) \Bigg|_{x=1}. \end{aligned}$$

By the assumption, this is equal to 0. Thus, the induction continues. This completes the proof of the Theorem. (Q.E.D.)

3. The Euler-Maclaurin Summation Formula

3.1 Introduction

Consider the following problem: For a given function $f(x)$, find a formula for the sum

$$S = f(M+1) + f(M+2) + f(M+3) + \cdots + f(N-1) + f(N),$$

where N and M are real numbers and $N - M$ is a positive integer.

By the Taylor series expansion, we have

$$f(k-1) - f(k) = -f'(k) + \frac{1}{2!}f''(k) - \frac{1}{3!}f'''(k) + \frac{1}{4!}f^{(4)}(k) - \frac{1}{5!}f^{(5)}(k)\cdots,$$

or

$$f'(k) = f(k) - f(k-1) + \frac{1}{2!}f''(k) - \frac{1}{3!}f'''(k) + \frac{1}{4!}f^{(4)}(k) - \frac{1}{5!}f^{(5)}(k)\cdots.$$

By replacing f by its primitive (again denoted by f) and taking summation from $k = M+1$ to $k = N$, we get

$$\sum_{k=M+1}^{N} f(k) := \sum_{k=1}^{N-M} f(M+k)$$
$$= \int_M^N f(x)dx + \frac{1}{2!}\sum_{k=M+1}^{N} f'(k) - \frac{1}{3!}\sum_{k=M+1}^{N} f''(k) + \frac{1}{4!}\sum_{k=M+1}^{N} f'''(k) - \cdots.$$

Applying this method for f', we have

$$\sum_{k=M+1}^{N} f'(k) = f(N) - f(M) + \frac{1}{2!}\sum_{k=M+1}^{N} f''(k) - \frac{1}{3!}\sum_{k=M+1}^{N} f'''(k) + \frac{1}{4!}\sum_{k=M+1}^{N} f^{(4)}(k) + \cdots.$$

Substituting this formula in the above equation, we have

$$\sum_{k=M+1}^{N} f(k)$$
$$= \int_M^N f(x)dx + \frac{1}{2!}(f(N) - f(M)) - \left(\frac{1}{2!2!} - \frac{1}{3!}\right)\sum_{k=M+1}^{N} f''(k) + \left(\frac{1}{2!3!} - \frac{1}{4!}\right)\sum_{k=M+1}^{N} f'''(k) + \cdots.$$

Continuing this method for higher derivatives of f, we have the Euler-Maclaurin summation formula:

$$\sum_{k=M+1}^{N} f(k) = \int_M^N f(x)dx + \frac{1}{2!}(f(N) - f(M)) + \frac{1}{12}(f'(N) - f'(M)) - \frac{1}{720}(f'''(N) - f'''(M)) + \cdots.$$

3.2 Bernoulli numbers and Bernoulli polynomials

In this section, we prepare some notions necessary for more precise formulation of the Euler-Maclaurin summation formula.

Definition. For each non-negative integer n, Bernoulli number, denoted by B_n, is defined by the following generating function:

$$\frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n.$$

The radius of convergence of the above power series is equal to 2π.

Similarly, Bernoulli polynomial, denoted by $B_n(x)$, is defined by means of:

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n(x)}{n!} t^n.$$

Remarks.

(i) From the definition, Bernoulli numbers are the constant terms of Bernoulli polynomial, i.e. $B_n = B_n(0)$.

(ii) Bernoulli numbers and polynomials have the following properties:

$$B_0 = 1, \qquad B_1 = -\frac{1}{2}, \qquad \sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \ (\text{for } n \geqq 2).$$

$$B_{2n+1} = 0 \ (\text{for } n \geqq 1), \qquad B_n(0) = B_n(1) = B_n \ (\text{for } n \geqq 2).$$

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \ (\text{for } n \geqq 0).$$

$$B_0(x) = 1, \qquad B_n'(x) = nB_{n-1}(x) \ (\text{for } n \geqq 1), \qquad \int_0^1 B_n(x)dx = 0 \ (\text{for } n \geqq 1).$$

(iii) Bernoulli numbers can be calculated, for instance, by way of power series expansion.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{B_n}{n!} t^n &= \frac{t}{e^t - 1} = \frac{1}{1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots} \\
&= 1 - \left(\frac{t}{2!} + \frac{t^2}{3!} + \cdots\right) + \left(\frac{t}{2!} + \frac{t^2}{3!} + \cdots\right)^2 - \left(\frac{t}{2!} + \frac{t^2}{3!} + \cdots\right)^3 + \cdots \\
&= 1 - \frac{1}{2}t + \frac{1}{12}t^2 - \frac{1}{720}t^4 + \cdots .
\end{aligned}$$

Compare the coefficients of the above series and those appearing in the last equation in section 3.1.

Table.

n	Numerator of B_n	Denominator of B_n	Values of B_n
0	1	1	1.00000
1	-1	2	-0.50000
2	1	6	0.16667
4	-1	30	-0.03333
6	1	42	0.02381
8	-1	30	-0.03333
10	5	66	0.07576
12	- 691	2730	-0.25311
14	7	6	1.16667
16	-3617	510	-7.09216
18	43867	798	54.97118
20	-174611	330	-529.12424

3.3 The Euler-Maclaurin summation formula

Theorem I.3.1. (The Euler-Maclaurin summation formula) *If a function $f(x)$ is differentiable for n times and $f^{(n)}(x)$ is continuous on $[M, N]$, where $N - M$ is a positive integer, then*

$$\sum_{k=0}^{N-M} f(M+k) = \int_M^N f(x)dx + \frac{1}{2}(f(N) + f(M)) + \sum_{j=2}^{n} \frac{B_j}{j!} f^{(j-1)}(x)\Big|_{x=M}^{N} + T_n.$$

The reminder term is given by

$$T_n = \frac{(-1)^{n-1}}{n!} \int_M^N \tilde{B}_n(x) f^{(n)}(x)dx,$$

where $\tilde{B}_n(x) = B_n(x - [x])$ is the periodical extension of $B_n(x)$ on $[0,1]$ over $\mathbb{R}$, and $[x]$ denotes the largest integer that does not exceed x.

Proof. Consider the following integral:

$$I_k = \int_k^{k+1} \tilde{B}_1(x) f'(x)dx = \int_0^1 B_1(x) f'(x+k)dx = \int_0^1 \left(x - \frac{1}{2}\right) f'(x+k)dx.$$

We apply the integration by parts in two different manners.

On the one hand,

$$\begin{aligned}I_k &= \left(x-\frac{1}{2}\right)f(x+k)\Big|_{x=0}^{1} - \int_0^1 f(x+k)dx \\ &= \frac{1}{2}(f(k+1)+f(k)) - \int_0^1 f(x+k)dx \\ &= \frac{1}{2}(f(k+1)+f(k)) - \int_k^{k+1} f(x)dx.\end{aligned}$$

On the other hand,

$$\begin{aligned}I_k &= \int_0^1 B_1(x)f'(x+k)dx = \frac{1}{2}B_2(x)f'(x+k)\Big|_{x=0}^{1} - \frac{1}{2}\int_0^1 B_2(x)f''(x+k)dx \\ &= \frac{1}{2}B_2(1)f'(k+1) - \frac{1}{2}B_2(0)f'(k) - \frac{1}{2}\int_0^1 B_2(x)f''(x+k)dx \\ &= \frac{1}{2}B_2f'(x)\Big|_{x=k}^{k+1} - \frac{1}{2}\int_k^{k+1}\tilde{B}_2(x)f''(x)dx.\end{aligned}$$

Applying this method recursively, we have

$$I_k = \sum_{j=2}^{n}(-1)^j\frac{B_j}{j!}f^{(j-1)}(x)\Big|_{x=k}^{k+1} + \frac{(-1)^{n-1}}{n!}\int_k^{k+1}\tilde{B}_n(x)f^{(n)}(x)dx.$$

In the summation term above, as $B_3 = B_5 = B_7 = \cdots = 0$, the factor $(-1)^j$ for each non-zero term is equal to 1. Therefore, we have proved

$$f(k) = \int_k^{k+1} f(x)dx - \frac{1}{2}(f(k+1)-f(k)) + \sum_{j=2}^{n}\frac{B_j}{j!}f^{(j-1)}(x)\Big|_{x=k}^{k+1} + \frac{(-1)^{n-1}}{n!}\int_k^{k+1}\tilde{B}_n(x)f^{(n)}(x)dx.$$

Summing up this formula from $k = M$ to $N-1$, and adding $f(N)$ to both sides, we obtain the required statement. (Q.E.D.)

3.4 Estimation of the reminder of the Euler-Maclaurin summation formula

To estimate the reminder of the Euler-Maclaurin summation formula, T_n, we quote the following formula[1]:

Proposition (Euler).

$$B_{2n} = (-1)^{n-1}\frac{2(2n)!}{(2\pi)^{2n}}\sum_{k=1}^{\infty}\frac{1}{k^{2n}} \qquad (n \geqq 1) - .$$

[1]For the proof, see Bourbaki or Dieudonné in the references.

From this,

$$|B_{2n}| = \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}} \leqq \frac{2(2n)!}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{2(2n)!}{(2\pi)^{2n}} \cdot \frac{\pi^2}{6}.$$

Hence, for $0 \leqq x \leqq 1$, we have

$$\begin{aligned}
|B_n(x)| &\leqq \left| \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k} \right| \leqq \sum_{k=0}^{n} \binom{n}{k} |B_k| \leqq \frac{\pi^2}{3} \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(2\pi)^k} \\
&= \frac{\pi^2}{3} \sum_{k+l=n,\ k\geq 0,\ l\geq 0} \frac{n!}{k!l!} \cdot \frac{k!}{(2\pi)^{n-l}} = \frac{\pi^2}{3} \frac{n!}{(2\pi)^n} \sum_{l=0}^{n} \frac{(2\pi)^l}{l!} \leqq \frac{\pi^2}{3} \frac{n!}{(2\pi)^n} e^{2\pi}.
\end{aligned}$$

Hence,

$$|T_n| \leqq \frac{1}{n!} \max_{x\in[0,1]} |B_n(x)| \int_M^N |f^{(n)}(x)| dx \leqq \frac{\pi^2 e^{2\pi}(N-M)}{3} \frac{\|f^{(n)}\|_\infty}{(2\pi)^n},$$

where $\|f^{(n)}\|_\infty = \max_{x\in[M,N]} |f^{(n)}(x)|$.

Therefore, if a function f is infinitely differentiable and satisfies the condition:

[$*$] $\quad \|f^{(n)}\| = o((2\pi)^n)$,

then $|T_n| \to 0$ (as $n \to \infty$).

In addition, the condition [$*$] implies that $\|f^{(n)}\|_\infty = o(n!(2\pi)^n)$. Thus, f satisfies the condition [$\#\#'$] in I.1.3; therefore, f is analytic on (M, N). (Note that, in general, $u_n = o(w_n)$ implies $u_n = O(w_n)$).

Further, a sufficient condition for [$*$] is

[$**$] $\quad$ There exists $0 < \alpha < 2\pi$ such that $\|f^{(n)}\|_\infty = O(\alpha^n)$.

This is equivalent to:

[$**'$] $\quad$ There exist $A > 0$ and $0 < \alpha < 2\pi$ such that $|f^{(n)}(x)| \leqq A\alpha^n$ (for all x and for all n);

and further,

[$**''$] $\quad \limsup_{n\to\infty} \left(\|f^{(n)}\|_\infty\right)^{\frac{1}{n}} < 2\pi \qquad$ (Note that $\lim_{n\to\infty} A^{\frac{1}{n}} = 1$).

If f satisfies one of the above three conditions, the radius of convergence of f is ∞. Hence, in this case, f can be extended analytically on $\mathbb{R}$ and $|f(x)| \leqq Ae^{\alpha|x|}$ on $\mathbb{R}$.

Remark. The Euler-Maclaurin summation formula shows that a summation can be calculated by one integration and higher orders of differentiation. We can give another interpretation of this formula in terms of differential and finite difference operators.

Noting that $B_1 = -\frac{1}{2}$, we can write the Euler-Maclaurin summation formula as follows:

$$\sum_{k=M}^{N-1} f(k) = \int_M^N f(x)dx + \sum_{j=1}^{\infty} \frac{B_j}{j!} f^{(j-1)}(x)\Big|_{x=M}^{N}.$$

This equation can be written as

$$\Delta^{-1} f(k)\Big|_{k=M}^{N} = \left(\sum_{j=0}^{\infty} \frac{B_j}{j!}\partial^{(j-1)}\right) \cdot f(x)\Big|_{x=M}^{N} = \left(\frac{1}{e^{\partial}-1}\right) \cdot f(x)\Big|_{x=M}^{N}.$$

In the above calculation, $\partial^{-1} f(x)$ denotes a primitive of f.

By comparing these results, the Euler-Maclaurin summation formula can be symbolically written as: $\Delta^{-1} = 1/(e^{\partial}-1)$. This can also be considered as a variant of the formal relation: $e^{\partial} = 1 + \Delta$.

3.5 The Woolhouse summation formula

Theorem I.3.2. (The Woolhouse summation formula) *If f satisfies the condition [**] (or [**′] or [**″]) in section 3.4, then the following formula holds for any integers m and n and for any real number a:*

$$\frac{1}{m}\sum_{k=0}^{mn} f\left(a+\frac{k}{m}\right) = \sum_{k=0}^{n} f(a+k) - \frac{m-1}{2m}(f(a)+f(a+n)) - \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!}\frac{m^{2j}-1}{m^{2j}} f^{(2j-1)}(a+x)\Big|_{x=0}^{n}.$$

Proof. From the Euler-Maclaurin summation formula, we have

$$\sum_{k=M}^{N} g(k) = \int_M^N g(x)dx + \frac{1}{2}(g(M)+g(N)) + \sum_{j=1}^{\infty} \frac{B_{2j}}{(2j)!} g^{(2j-1)}(x)\Big|_{x=M}^{N}$$

with $g(x) = f(a + \frac{x}{m})$, $M = 0$, $N = mn$, then

$$\begin{aligned}&\frac{1}{m}\sum_{i=0}^{mn} f\left(a+\frac{i}{m}\right)\\ =&\frac{1}{m}\int_0^{mn} f\left(a+\frac{x}{m}\right)dx + \frac{f(a)+f(a+n)}{2m} + \frac{1}{m}\sum_{j=1}^{\infty}\frac{B_{2j}}{(2j)!}\cdot\left(\frac{d}{dx}\right)^{2j-1} f\left(a+\frac{x}{m}\right)\Big|_{x=0}^{mn}.\end{aligned}$$

By changing variable, the first term on the right-hand side is written:

$$\frac{1}{m}\int_0^{mn} f\left(a+\frac{x}{m}\right)dx = \int_0^n f(a+t)dt.$$

Hence,

$$\frac{1}{m}\sum_{i=0}^{mn} f\left(a+\frac{i}{m}\right) = \int_0^n f(a+t)dt + \frac{f(a)+f(a+n)}{2m} + \sum_{j=1}^{\infty}\frac{B_{2j}}{(2j)!}\cdot\frac{1}{m^{2j}} f^{(2j-1)}(a+t)\Big|_{t=0}^{n}.$$

By applying again the Euler-Maclaurin summation formula to $g(x) = f(a+x)$, with $M = 0$, $N = n$,

$$\sum_{i=0}^{n} f(a+i) = \int_0^n f(a+t)dt + \frac{f(a)+f(a+n)}{2} + \sum_{j=1}^{\infty}\frac{B_{2j}}{(2j)!}\cdot f^{(2j-1)}(a+t)\Big|_{t=0}^{n}.$$

By substituting this into the above equation, we have the required formula.(Q.E.D.)

Remark 1. In the Woolhouse summation formula, if we tend m to infinity, then

$$\frac{1}{m}\sum_{k=0}^{mn} f\left(a+\frac{k}{m}\right) \to \int_0^n f(a+x)dx.$$

Thus, the Woolhouse summation formula reproduces the Euler-Maclaurin summation formula.

Remark 2. Recall the definition of the $\frac{1}{m}$ -difference operator: $\Delta^{(m)}\cdot f(x) = m\left(f\left(x+\frac{1}{m}\right) - f(x)\right)$.

Consider the inverse operation. If $\dfrac{\Delta^{(m)}}{m}\cdot F(x) = f\left(\dfrac{x}{m}\right)$ then

$$\frac{1}{m}\sum_{x=0}^{mn-1} f\left(\frac{x}{m}\right) = F(x)\Big|_{x=0}^{n} = (\Delta^{(m)})^{-1} f\left(\frac{x}{m}\right)\Big|_{x=0}^{n}.$$

Note that

$$\lim_{m\to\infty}(\Delta^{(m)})^{-1} f\left(\frac{x}{m}\right)\Big|_{x=0}^{n} = \int_0^n f(x)\ dx = \partial^{-1} f(x)\Big|_{x=0}^{n}.$$

The Woolhouse summation formula is written as:

$$\frac{1}{m}\sum_{k=0}^{mn-1} f\left(a+\frac{k}{m}\right) = \sum_{k=0}^{n-1} f(a+k) - \sum_{j=0}^{\infty}\frac{B_j}{j!} f^{(j-1)}(a+x)\Big|_{x=0}^{n} + \sum_{j=0}^{\infty}\frac{B_j}{j!}\frac{1}{m^j} f^{(j-1)}(a+x)\Big|_{x=0}^{n}.$$

Formally, this is equivalent to

$$(\Delta^{(m)})^{-1} = \Delta^{-1} - \frac{1}{e^{\partial}-1} + \frac{1}{m}\cdot\frac{1}{e^{\partial/m}-1}.$$

Again, this is justified by the formal relations:

$$\Delta^{-1} = \frac{1}{e^{\partial} - 1}; \qquad (\Delta^{(m)})^{-1} = \frac{1}{m} \cdot \frac{1}{e^{\partial/m} - 1}.$$

These relations are derived by formal operations from:

$$e^{\partial} = 1 + \Delta = \left(\frac{1 + \Delta^{(m)}}{m} \right)^{m}.$$

4. Concluding Summary

To conclude, we summarise all the formulae clarified in this note in terms of formal notation.

Definition of the operators and their properties:

$$\Delta^{(m)} \cdot f(x) = m \left(f\left(x + \frac{1}{m} \right) - f(x) \right).$$

Note that $\Delta^{(1)} = \Delta,\ \lim_{m\to\infty} \Delta^{(m)} = \partial,\ \lim_{m\to\infty} (\Delta^{(m)})^{-1} = \int dx = \partial^{-1}.$

The Taylor series expansion, Newton's interpolation formula:

$$e^{h\partial} = (1 + \Delta)^{h} = \left(1 + \frac{\Delta^{(m)}}{m} \right)^{mh}.$$

Numerical differentiation (using Newton's formula):

$$\partial = \log(1 + \Delta).$$

The Euler-Maclaurin summation formula:

$$\Delta^{-1} = \frac{1}{e^{\partial} - 1}; \qquad (\Delta^{(m)})^{-1} = \frac{1}{m} \cdot \frac{1}{e^{\partial/m} - 1}.$$

The Woolhouse summation formula:

$$(\Delta^{(m)})^{-1} = \Delta^{-1} - \frac{1}{e^{\partial} - 1} + \frac{1}{m} \cdot \frac{1}{e^{\partial/m} - 1}.$$

Part II. Applications in Numerical Analysis

1. Interpolation

1.1 Polynomial interpolation

Definition. (Divided difference) Suppose that the values of a function $f(x)$ are given at points $x = x_0, x_1, x_2, \ldots, x_n (x_i \neq x_j$, for $i \neq j)$, then we define $f[x_0, x_1, x_2, \ldots, x_n]$ recursively by:

1° $f[x_0] = f(x_0)$.

2° $f[x_0, x_1, \ldots, x_{n-1}, x_n] = -\dfrac{f[x_0, x_1, \ldots, x_{n-1}, \widehat{x_n}] - f[x_0, x_1, \ldots, \widehat{x_{n-1}}, x_n]}{x_n - x_{n-1}}$

(for $n \geqq 1$). Here, the notation $\widehat{x_i}$ means that the term x_i is omitted.

Theorem II.1.1. (Generalized Newton's formula) *For* $n \geqq 1$,

$$f(x) = \sum_{i=0}^{n} \left(\prod_{k=0}^{n} (x - x_k) \right) \cdot f[x_0, x_1, \ldots, x_i] + \left(\prod_{k=0}^{n} (x - x_k) \right) \cdot f[x_0, x_1, \ldots, x_n, x].$$

Proof. For $n = 1$, by definition, $f(x) = f(x_0) + (x - x_0) f[x_0, x]$.

For $n = 2$, by the definition of divided differences,

$$f[x_0, x] = f[x_0, x_1] + f[x_0, x_1, x] \cdot (x - x_1).$$

Substituting this into the above, we have

$$f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x].$$

For $n = 3$, by the definition of divided differences,

$$f[x_0, x_1, x] = f[x_0, x_1, x_2] + f[x_0, x_1, x_2, x] \cdot (x - x_2).$$

Thus,

$$\begin{aligned} f(x) = f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ + (x - x_0)(x - x_1)(x - x_2) f[x_0, x_1, x_2, x]. \end{aligned}$$

By continuing this method, we arrive at the general statement. (Q.E.D.)

Concerning the interpolation with polynomial, we have the following general formula.

Theorem II.1.2. (Lagrange's interpolation formula) *Let $(x_0, a_0), (x_1, a_1), \ldots, (x_n, a_n)$ be $n+1$ points ($x_i \neq x_j$ for $i \neq j$). There exists a unique polynomial $L(x)$ of degree n that passes all these $n+1$ points. It is written in the following form:*

$$L(x) = \sum_{i=0}^{n} a_i \prod_{k \neq i} \frac{x - x_k}{x_i - x_k} = \sum_{i=0}^{n} \frac{a_i W(x)}{(x - x_i) W'(x_i)}, \qquad \textit{where } W(x) = \prod_{k=0}^{n} (x - x_k).$$

Proof. Uniqueness is clear. Direct calculation will ensure that $L(x_i) = a_i$ and the other results. (Q.E.D.)

Theorem II.1.3. *Under the same notation of Theorem II.1.1, put*

$$\varphi(x) = \sum_{i=0}^{n} \left(\prod_{k=0}^{i-1} (x - x_k) \right) \cdot f[x_0, x_1, \ldots, x_i].$$

Then, $\varphi(x)$ is the polynomial of degree n that passes $n+1$ points $(x_i, f(x_i))$ (for $i = 0, 1, \ldots, n$). In particular,

$$\varphi(x) = \sum_{i=0}^{n} \left(\prod_{k=0}^{i-1} (x - x_k) \right) \cdot f[x_0, x_1, \ldots, x_i] = \sum_{i=0}^{n} f(x_i) \prod_{k \neq i} \frac{x - x_k}{x_i - x_k} = \sum_{i=0}^{n} \frac{f(x_i) W(x)}{(x - x_i) W'(x_i)}.$$

Proof. By Theorem II.1.1,

$$\varphi(x) = f(x) - \left(\prod_{k=0}^{i-1} (x - x_k) \right) \cdot f[x_0, x_1, \ldots, x_n, x].$$

We observe that (i) $\varphi(x)$ is a polynomial of degree n; and, (ii) $\varphi(x_i) = f(x_i)$ (for $i = 0, 1, \ldots, n$). Hence, by the uniqueness of the interpolation polynomial, the first statement follows. By combining this with Lagrange's interpolation formula, the second statement is shown. (Q.E.D.)

Corollary 1.

$$f[x_0, x_1, \ldots, x_n] = \sum_{i=0}^{n} f(x_i) \frac{1}{\prod_{k \neq i,\ 0 \leq k \leq n} (x_i - x_k)}.$$

Proof. Compare the coefficient of the leading term of the last equation in the Theorem. (Q.E.D.)

Corollary 2. $f[x_0, x_1, \ldots, x_n]$ *is invariant under any permutation of* $\{x_i\}$.

Proof. From Corollary 1, it is evident. (Q.E.D.)

Corollary 3.

$$f[x, x+1, \ldots, x+n] = \frac{1}{n!} \Delta^n f(x).$$

Proof. From Corollary 1,

$$f[x, x+1, \ldots, x+n] = \sum_{i=0}^{n} f(x+i) \frac{1}{\prod_{k \neq i,\ 0 \leq k \leq n} (i-k)}$$
$$= \frac{1}{n!} \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) \cdot (-1)^{n-i} f(x+i) = \frac{1}{n!} \Delta^n f(x).$$

Here, we used the following fact:

$$\frac{n!}{\prod_{k \neq i,\ 0 \leq k \leq n} (i-k)} = \left(\begin{array}{c} n \\ i \end{array} \right) \cdot (-1)^{n-i}.$$

(Q.E.D.)

Concerning the estimation of error, we have

Theorem II.1.4. *Using the same notation as Theorem II.1.3, if each x_i belongs to (a,b) and f is differentiable for $n+1$ times and $f^{(n+1)}$ is continuous on (a,b), then for each $x \in (a,b)$, there exists $\xi \in (a,b)$ such that*

$$f(x) - \varphi(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot W(x).$$

Proof. Suppose x is distinct from any x_i (otherwise it is trivial). Put

$$\phi(t) = f(t) - \varphi(t) - (f(x) - \varphi(x)) \frac{W(t)}{W(x)}.$$

Since $\phi(t)$ has $n+2$ distinct zeros $t = x, x_0, x_1, x_2, \ldots, x_n$, applying Rolle's theorem for $n+1$ times, there exists $\xi \in (a,b)$ such that

$$0 = \phi^{(n+1)}(\xi) = f^{(n+1)}(\xi) - \varphi^{(n+1)}(\xi) - (f(x) - \varphi(x)) \frac{W^{(n+1)}(\xi)}{W(x)}.$$

The required statement will follow by noting that $\varphi^{(n+1)}(\xi) = 0$ and $W^{(n+1)}(\xi) = (n+1)!$. (Q.E.D.)

Remark 1. (knowledge of calculation of residues is assumed). Suppose that f is extended to a holomorphic function on a complex domain D containing $[a,b]$. Then, in the interior of D, we draw a closed curve C surrounding $[a,b]$. We then have

$$\frac{1}{2\pi i} \int_C \frac{W(z) - W(x)}{(z-x)W(z)} f(z) dz = \sum_{i=0}^{n} \mathrm{Res}_{z=x_i} \left[\frac{W(z) - W(x)}{(z-x)W(z)} f(z) dz \right]$$
$$= \sum_{i=0}^{n} \lim_{z \to x_i} \left[(z - x_i) \frac{W(z) - W(x)}{(z-x)W(z)} f(z) \right]$$
$$= \sum_{i=0}^{n} \lim_{z \to x_i} \left[\frac{W(z) - W(x)}{z - x} \cdot \frac{f(z)}{\frac{W(z) - W(x_i)}{z - x_i}} \right]$$
$$= \sum_{i=0}^{n} \frac{W(x)}{(x - x_i)} \cdot \frac{f(x_i)}{W'(x_i)} = \varphi(x).$$

Thus,

$$f(x)-\varphi(x)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-x}dz-\frac{1}{2\pi i}\int_C \frac{W(z)-W(x)}{(z-x)W(z)}f(z)\,dz=\frac{1}{2\pi i}\int_C \frac{W(x)f(z)}{(z-x)W(z)}\,dz.$$

Remark 2. Given a function f on an interval I. Let φ_n be an interpolating polynomial at equally spaced $(n+1)$ nodes on I. Even if f is analytic, φ_n may not necessarily converge uniformly to f on I, as n increases to infinity. A well-known example, called Runge's phenomenon, is the case in which $f(x)=1/(x^2+\alpha^2)$ (for $\alpha>0$) and $I=[-1,1]$. In this case, $|f(1)-\varphi_n(1)|$ does not necessarily converge to zero. Furthermore, for a sufficiently small α, this absolute value diverges exponentially as n tends to infinity. However, concerning this phenomenon, two facts are known[1]:

(i) In the above example, if non-equally spaced points $x_i=\cos((2i+1)\pi/2(n+1))$ are taken, then the interpolant polynomial will converge uniformly to f. (The Chebyshev interpolation).

(ii) Consider $f(z)$ as a function of a complex variable. If $f(z)$ has no singularity in $D=\{z\in\mathbb{C};\ (z+1)^{z+1}/(z-1)^{z-1}>4e^{\pi|\mathrm{Im}\ z|}\}$, then the polynomial interpolant with equally spaced nodes will converge uniformly to f. (In the above example, if α is small the poles of $f(z)$ $(z=\pm\alpha i)$ are in D).

1.2 Interpolation formulae

Given $(n+1)$ points, the application of Theorem II.1.3 to particular cases will yield the following variant of interpolation formulae.

(i) Newton's forward formula:

$$p_{NF}(x)=\sum_{k=0}^{n}\binom{x}{k}\Delta^k p(0).$$

(ii) Newton's backward formula:

$$p_{NB}(x)=\sum_{k=0}^{n}\binom{x+k-1}{k}\Delta^k p(-k).$$

(iii) Gauss's forward formula:

$$p_{GF}(x)=\sum_{k=0}^{[n/2]}(\phi_k^{GFe}\Delta^{2k}p(-k)+\phi_k^{GFo}(x)\Delta^{2k+1}p(-k)),$$

[1] For details, see Sugihara and Murota, and Iri and Fujino in the references.

where

$$\phi_k^{GFe}(x) = \left(\begin{array}{c} x+k-1 \\ 2k \end{array} \right) = \frac{x(x^2-1)\cdots(x^2-(k-1)^2)(x-k)}{(2k)!},$$

$$\phi_k^{GFo}(x) = \left(\begin{array}{c} x+k \\ 2k+1 \end{array} \right) = \frac{x(x^2-1)\cdots(x^2-k^2)}{(2k+1)!}.$$

(Note: If $n = 2m$ then the last term is zero, because $\Delta^{2m+1}p(x) = \Delta^{n+1}p(x) = 0$).

(iv) Gauss's backward formula:

$$p_{GB}(x) = \sum_{k=0}^{[n/2]} (\phi_k^{GBe}\Delta^{2k}p(-k) + \phi_k^{GBo}(x)\Delta^{2k+1}p(-k-1)),$$

where

$$\phi_k^{GBe}(x) = \left(\begin{array}{c} x+k \\ 2k \end{array} \right) = \frac{x(x^2-1)\cdots(x^2-(k-1)^2)(x+k)}{(2k)!},$$

$$\phi_k^{GBo}(x) = \left(\begin{array}{c} x+k \\ 2k+1 \end{array} \right) = \frac{x(x^2-1)\cdots(x^2-k^2)}{(2k+1)!}.$$

(v) Stirling's formula ($n = 2m$):

$$p_S(x) = \sum_{k=0}^{m} \phi_k^{Se}(x)\delta^{2k}p(0) + \sum_{k=0}^{m-1} \phi_k^{So}(x)\mu\delta^{2k+1}p(0),$$

where, $\delta f(x) = f(x+\frac{1}{2}) - f(x-\frac{1}{2})$ (Central difference), and $\mu f(x) = \frac{1}{2}[f(x+\frac{1}{2}) + f(x-\frac{1}{2})]$ (Mean difference), and

$$\phi_k^{Se}(x) = \frac{x}{2k}\left(\begin{array}{c} x+k-1 \\ 2k-1 \end{array} \right) = \frac{x^2(x^2-1)\cdots(x^2-(k-1)^2)}{(2k)!},$$

$$\phi_k^{So}(x) = \left(\begin{array}{c} x+k \\ 2k+1 \end{array} \right) = \frac{x(x^2-1)\cdots(x^2-k^2)}{(2k+1)!}.$$

Further,

$$p_S(x) = \sum_{k=0}^{m} \rho_k^e(x)[p(k)+p(-k)] + \sum_{k=1}^{m} \rho_k^o(x)[p(k)-p(-k)],$$

where, $\rho_k^e(x)$ is an even function of x; and $\rho_k^o(x)$, an odd function of x.

Proof. (i) Applying Theorem II.1.3, with $x_0 = 0, x_1 = 1, x_2 = 2, \ldots, x_n = n$, we have

$$p(x) = \sum_{i=0}^{n}\left(\prod_{k=0}^{i-1}(x-k)\right)\cdot p[0,1,\ldots,i] = \sum_{i=0}^{n}\left(\prod_{k=0}^{i-1}(x-k)\right)\cdot\frac{1}{i!}\Delta^i p(0) = \sum_{i=0}^{n}\left(\begin{array}{c} x \\ i \end{array} \right)\Delta^i p(0).$$

(ii) Applying Theorem II.1.3, with $x_0 = 0, x_1 = -1, x_2 = -2, \ldots, x_n = -n$, we have

$$\begin{aligned} p(x) &= \sum_{i=0}^{n}\left(\prod_{k=0}^{i-1}(x-k)\right)\cdot p[0,-1,\ldots,-i] = \sum_{i=0}^{n}\left(\prod_{k=0}^{i-1}(x-k)\right)\cdot p[-i,\ldots,-1,0] \\ &= \sum_{i=0}^{n}\left(\prod_{k=0}^{i-1}(x-k)\right)\cdot\frac{1}{i!}\Delta^i p(-i) = \sum_{i=0}^{n}\left(\begin{array}{c} x+i-1 \\ i \end{array} \right)\Delta^i p(-i). \end{aligned}$$

(iii) Applying Theorem II.1.3, with $x_0 = 0, x_1 = 1, x_2 = -1, \ldots, x_{2i} = i, x_{2i+1} = -i, \ldots$, we have

$$p(x) = \sum_{i \geq 0} \left[(x-i)x \prod_{k=0}^{i-1} (x-k)(x+k) \cdot p[0,1,-1,\ldots,i,-i] \right. \\ \left. + x \prod_{k=0}^{i} (x-k)(x+k) \cdot p[0,1,-1,\ldots,i,-i,i+1] \right],$$

where

$$p[0,1,-1,\ldots,i,-i] = p[-i,\ldots,-1,0,1,\ldots,i] = \frac{1}{(2i)!}\Delta^{2i}p(-i),$$
$$p[0,1,-1,\ldots,i,-i,i+1] = p[-i,\ldots,-1,0,1,\ldots,i,i+1] = \frac{1}{(2i+1)!}\Delta^{2i+1}p(-i).$$

(iv) Applying Theorem II.1.3, with $x_0 = 0, x_1 = -1, x_2 = 1, \ldots, x_{2i} = -i, x_{2i+1} = i, \ldots$, we have

$$p(x) = \sum_{i \geq 0} \left[(x-i)x \prod_{k=0}^{i-1} (x+k)(x-k) \cdot p[0,-1,1,\ldots,-i,i] \right. \\ \left. + x \prod_{k=0}^{i} (x-k)(x+k) \cdot p[0,-1,1,\ldots,-i,i,-(i+1)] \right],$$

where

$$p[0,-1,1,\ldots,-i,i] = p[-i,\ldots,-1,0,1,\ldots,i] = \frac{1}{(2i)!}\Delta^{2i}p(-i),$$
$$p[0,-1,1,\ldots,-i,i,-(i+1)] = p[-(i+1),-i,\ldots,-1,0,1,\ldots,i] = \frac{1}{(2i+1)!}\Delta^{2i+1}p(-i-1).$$

(v) Taking the termwise average of formulae (iii) and (iv), we have

$$\text{Even-terms} = \frac{1}{2}\left[\binom{x+k-1}{2k} + \binom{x+k}{2k} \right] \cdot \Delta^{2k}p(-k) = \frac{x}{2k}\binom{x+k-1}{2k-1}\delta^{2k}p(0).$$

(Note that $\delta^{2k} = \Delta^{2k} \cdot (1+\Delta)^{-k}$).

Further,

$$\begin{aligned} \text{Odd-terms} &= \binom{x+k}{2k+1} \cdot \frac{1}{2}(\Delta^{2k}\Delta p(-k) + \Delta^{2k}\Delta p(-k-1)) \\ &= \binom{x+k}{2k+1} \cdot \frac{1}{2}\Delta^{2k}((p(-k+1) - p(-k-1))) \\ &= \binom{x+k}{2k+1} \cdot \Delta^{2k}\mu\delta \cdot p(-k) \\ &= \binom{x+k}{2k+1} \cdot \mu\delta\delta^{2k} \cdot p(0). \end{aligned}$$

To show the alternative formula, we first note that

$$\begin{aligned}\delta^{2k}\cdot p(0)=\Delta^{2k}\cdot p(-k)&=\sum_{j=0}^{2k}\binom{2k}{j}(-1)^{2k}p(-k+j)\\&=\sum_{j=0}^{k}(-1)^{2k}\binom{2k}{k+j}p(j)+\sum_{j=1}^{k}(-1)^{2k}\binom{2k}{k-j}p(-j)\\&=\sum_{j=0}^{k}\alpha_{k,j}[p(j)+p(-j)],\end{aligned}$$

where

$$\alpha_{k,j}=(-1)^{2k}\binom{2k}{k+j}=(-1)^{2k}\binom{2k}{k-j}=\alpha_{k,-j}\ (j>0),\ \ \alpha_{k,0}=\frac{(-1)^{2k}}{2}\binom{2k}{k}.$$

Further, noting that $\mu\delta f(x)=\frac{1}{2}[f(x+1)-f(x-1)]$, we have

$$\begin{aligned}(\mu\delta)\delta^{2k}\cdot p(0)&=\mu\delta\left(\sum_{j=0}^{k}\alpha_{k,j}[p(j)-p(-j)]\right)\\&=\sum_{j=0}^{k}\frac{1}{2}\alpha_{k,j}[p(j+1)+p(j-1)+p(-j+1)+p(-j-1)]\\&=\sum_{j=0}^{k}\frac{1}{2}\alpha_{k,j}[p(j+1)-p(-(j+1))]+[p(j-1)-p(-(-j-1))]\\&=\sum_{j=1}^{k+1}\frac{1}{2}\beta_{k,j}[p(j)-p(-j)],\end{aligned}$$

where

$$\beta_{k,1}=\frac{1}{2}\alpha_{k,1}\ ;\ \beta_{k,j}=\frac{1}{2}(\alpha_{k,j-1}+\alpha_{k,j+1})\ (2\leqq j\leqq k-1)\ ;\ \beta_{k,j}=\frac{1}{2}\alpha_{k,j-1}\ \ (j=k,k+1).$$

Therefore,

$$\begin{aligned}\sum_{k=0}^{m}\phi_k^{Se}(x)\delta^{2k}p(0)&=\sum_{k=0}^{m}\phi_k^{Se}(x)\sum_{j=0}^{k}\alpha_{k,j}[p(j)+p(-j)]\\&=\sum_{j=0}^{m}\left(\sum_{k=0}^{j}\alpha_{k,j}\phi_k^{Se}(x)\right)[p(j)+p(-j)]=\sum_{j=0}^{m}\rho_j^{e}(x)[p(j)+p(-j)]\end{aligned}$$

and,

$$\begin{aligned}\sum_{k=0}^{m-1}\phi_k^{So}(x)\mu\delta^{2k+1}p(0)&=\sum_{k=0}^{m-1}\phi_k^{So}(x)\sum_{j=1}^{k+1}\beta_{k,j}[p(j)-p(-j)]\\&=\sum_{j=1}^{m}\left(\sum_{k=j-1}^{m-1}\beta_{k,j}\phi_k^{So}(x)\right)[p(j)-p(-j)]=\sum_{j=1}^{m}\rho_j^{m}(x)[p(j)-p(-j)].\end{aligned}$$

Since each $\phi_k^{Se}(x)$ is even, so is $\rho_k^e(x)$; and, each $\phi_k^{So}(x)$ is odd, so is $\rho_k^o(x)$.(Q.E.D.)

2. Numerical Differentiation

Suppose the values of a function $f(x)$ are given at $n+1$ points $f(a+x_ih)$ (for $i = 0,1,\ldots,n; x_i \neq x_j$ if $i \neq j; h > 0$). To calculate the derivatives $f^{(r)}(a)$, the following two methods are considered.

(i) Find the polynomial $\varphi(x)$ that passes these points and approximate $f^{(r)}(a)$ by $\varphi^{(r)}(a)$ $(r = 0,1,\ldots,n)$.

(ii) Approximate $f^{(r)}(a)$ by choosing suitable $\gamma_{r,k}$ $(r,k = 0,1,\ldots,n)$ such that

$$\Gamma(f,r;x_k)(a) := \sum_{k=0}^{n} \gamma_{r,k} \cdot f(a+x_kh) = f^{(r)}(a) + O(h^{n+1}).$$

Theorem II.2.1. *In the above notation, for any given $(x_i, f(x_i))$ (for $i = 0,1,\ldots,n; x_i \neq x_j$ if $i \neq j$) and $h > 0$, the values of $\gamma_{r,k}$ $(0 \leqq r,k \leqq n)$ exist and are identical to the result under the first method - i.e., $\Gamma(f,r;x_i)(a) = \varphi^{(r)}(a)(r = 0,1,\ldots,n)$.*

Proof. By Taylor's formula,

$$\begin{aligned}\sum_{k=0}^{n} \gamma_{r,k} f(a+x_jh) &= \sum_{k=0}^{n} \gamma_{r,k} \left(\sum_{l=0}^{n} \frac{f^{(l)}(a)}{l!}(x_kh)^l \right) + O(h^{n+1}) \\ &= \sum_{l=0}^{n} \left(\sum_{l=0}^{n} \gamma_{r,k} \frac{(x_kh)^l}{l!} \right) f^{(l)}(a) + O(h^{n+1})\end{aligned}$$

Hence, the condition for $\gamma_{r,k}$ is

$$\sum_{k=0}^{n} \gamma_{r,k} \frac{(x_kh)^l}{l!} = \delta_{lr}, \qquad \text{where } \delta_{lr} = \begin{cases} 0 & (\text{if } l \neq r) \\ 1 & (\text{if } l = r) \end{cases}$$

If we define $(n+1) \times (n+1)$ matrices by $W_{rk} = \gamma_{r,k}$ and $H_{kl} = (x_kh)^l/l!$ (for $r,k,l = 0,1,\ldots,n$), then the above condition is written as $WH = I_{n+1}$.

On the other hand, from Theorem II.1.3,

$$\varphi(x) = \sum_{j=0}^{n} \omega_j(x) f(a+x_jh), \qquad \text{where } \omega_j(x) = \prod_{k \neq j} \frac{x - x_k}{x_j - x_k}.$$

In particular, $\omega_j(x)$ must satisfy $\varphi(a+x_kh) = \sum_j \omega_j(a+x_kh) f(a+x_jh) = f(a+x_kh)$, hence $\omega_j(a+x_kh) = \delta_{kj}$. By Taylor's formula,

$$\omega_j(a+x_kh) = \sum_{l=0}^{n} \omega_j^{(l)}(a) \frac{(x_kh)^l}{l!} = \delta_{kj}.$$

Therefore if we define $(n+1)\times(n+1)$ matrix by $\Omega_{lj} = \omega_j^{(l)}(a)$ (for $k,l = 0,1,\ldots,n$), then the above relation is written as $H\Omega = I_{n+1}$.

From $H\Omega = I_{n+1}$, H is non-singular and its inverse matrix is $H^{-1} = \Omega$. Therefore, the equation $WH = I_{n+1}$ has a unique solution $W = H^{-1} = \Omega$, i.e., $\gamma_{r,k} = \omega_k^{(r)}(a)$. Therefore,

$$\sum_{k=0}^{n} \gamma_{r,k} f(a + x_k h) = \sum_{k=0}^{n} \omega_k^{(r)}(a) f(a + x_k h) = \varphi^{(r)}(a).$$

(Q.E.D.)

Remark. The non-singularity of H can be shown directly. Vandermonde's matrix of degree N is defined by $V_{ij} = X_j^{i-1}$, where $X_j (j = 1,\ldots,N)$ are N indeterminates. The formula for the determinant of V is

$$\det V = \det(X_j^{j-1}) = \prod_{i<j}(X_i - X_j).$$

By using this formula and noting the multilinearity of determinant with respect to rows and columns,

$$\det H = \det\left(\frac{(x_k h)^l}{l!}\right) = \frac{h^{1+2+\cdots+n}}{\prod_{i=1}^n i!} \cdot \det(x_k^l) = \frac{h^{\frac{n(n+1)}{2}}}{\prod_{i=1}^n i!} \cdot \prod_{i<j}(x_i - x_j) \neq 0.$$

Example 1. Suppose that values of a function $f(x)$ are given at $x = t, t+1, t+2, \ldots, t+n$. Differentiating the interpolating polynomial $f_{NF}(x+t)$ obtained by Newton's forward formula at $x = 0$, we have

$$f'_{NF}(t) = \sum_{k\geq 0} \frac{d}{dx}\left(\begin{array}{c} x \\ k \end{array}\right)\Big|_{x=0} \Delta^k f(t) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} \Delta^k f(t).$$

If $f(x)$ is a polynomial of degree n, then $f(x) \equiv f_{NF}(x)$. This gives another proof of Theorem I.2.2.

Example 2. Suppose that values of a function $f(x)$ are given at $2m+1$ points $x_k = t + hk$ (for $k = -m, -(m-1), \ldots, 0, \ldots, m-1, m$). Put $\phi(x) = t + hx$ and apply the Stirling's formula for $p(x) = f(\phi(x))$. By differentiating the interpolating polynomial $p_S(x) = f_S(\phi(x))$ at $x = 0$, we have

$$p'_S(0) = \sum_{k\geq 0} \frac{d}{dx}\left(\begin{array}{c} x+k \\ 2k+1 \end{array}\right)\Big|_{x=0} \mu\delta^{2k+1} p(0) = \sum_{k\geq 0} \frac{(-1)^k (k!)^2}{(2k+1)!} \cdot \mu\delta^{2k+1} p(0).$$

Note that $p'_S(x) = f'_S(\phi(0))\phi'(0) = f'_S(t)h$ and that $\delta p(0) = \delta f(\phi(0)) = \delta_h f(t)$ and $\mu p(0) = \mu f(\phi(0)) = \mu_h f(t)$. Here, $\delta_h f(t) = f(t + \frac{h}{2}) - f(t - \frac{h}{2})$ (Central h-difference) and $\mu_h f(t) = \frac{1}{2}[f(t+\frac{h}{2}) + f(t-\frac{h}{2})]$ (Mean h-difference). (Also note that $\mu_h \delta_h f(t) = \frac{1}{2}[f(t+h) - f(t-h)]$). Hence, we have

$$f'_S(t)h = \sum_{k\geq 0} \frac{(-1)^k (k!)^2}{(2k+1)!} \cdot \mu_h \delta_h^{2k+1} f(t).$$

In the above formula, taking the first term only (three points approximation), we have

$$f_S'(t) = \frac{1}{h}\mu_h\delta_h \cdot f(t) = \frac{f(t+h)-f(t-h)}{2h} =: F_1(t,h).$$

This is called the central difference approximation formula.

By taking the first two terms only (five points approximation), we have

$$f_S'(t) = \frac{1}{h}\left(\mu_h\delta_h \cdot f(t) - \frac{1}{6}\mu_h\delta_h^3 \cdot f(t)\right) = \frac{8[f(t+h)+f(t-h)]-[f(t+2h)-f(t-2h)]}{12} =: F_2(t,h).$$

Remark. Concerning the central difference approximation : $F_1(x,h) = [f(x+h) - f(x-h)]/(2h)$, the truncation error is $F_1(x,h) - f'(x) = \frac{1}{6}h^2|f^{(3)}(x)| + O(h^4)$ and the rounding error is $(2\varepsilon)/(2h) = \varepsilon/h$ (assuming the rounding of $f(x+h)$ involves an error in the order of ε). Therefore, the optimal value of h that minimises the total error $E(h;\varepsilon,f) = \frac{1}{6}h^2 f^{(3)}(x) + \varepsilon/h$ is attained at $h_{\text{opt}} = [3\varepsilon/|f^{(3)}(x)|]^{\frac{1}{3}}$, and the error is $3^{-\frac{1}{3}}\varepsilon^{\frac{2}{3}}|f^{(3)}(x)|^{\frac{1}{3}} + 3^{\frac{1}{3}}6^{-1}\varepsilon^{\frac{1}{3}}|f^{(3)}(x)|^{\frac{2}{3}} = O(\varepsilon^{\frac{2}{3}}|f^{(3)}(x)|^{\frac{1}{3}})$.

More generally, the following fact holds. By Stirling's formula,

$$f_S(t+hx) = \sum_{k=0}^{m} \rho_k^e(x)[f(t+kh)+f(t-kh)] + \sum_{k=1}^{m} \rho_k^o(x)[f(t+kh)-f(t-kh)],$$

where $\rho_k^e(x)$ is an even function of x; and $\rho_k^o(x)$, an odd function of x. Hence, for $q \geqq 0$

$$p_S^{(2q)}(t)h^{2q} = \sum_{k=0}^{m} \rho_k^{e\ (2q)}(t)[f(t+kh)+f(t-kh)],$$

and

$$p_S^{(2q+1)}(t)h^{2q+1} = \sum_{k=1}^{m} \rho_k^{o\ (2q+1)}(t)[f(t+kh)-f(t-kh)].$$

Hence, if we write for $r \geqq 0$

$$p_S^{(r)}(t)h^r = \sum_{k=-m}^{m} \theta_{r,k} f(t+kh)$$

then, the coefficients satisfy the relation $\theta_{r,k} = (-1)^r \cdot \theta_{r,-k}$.

3. Numerical Integration

The following Newton-Cotes integration formula is used to calculate a definite integral $I = \displaystyle\int_a^b f(x)dx$ numerically.

(1) Given an integer $n \geqq 1$. Put $x_k = a + hk$ $(k = 0, 1, \ldots, n-1)$, where $h = (b-a)/n$.

(2) Find a polynomial of degree n that interpolates $\{(x_k, f(x_k)); k = 0, 1, \ldots, n\}$ (by using Lagrange's formula).

(3) Calculate the integration of the interpolating polynomial over $[a, b]$ (this integral is denoted by I_n).

Theorem II.3.1. (The Newton-Cotes integration formula) *For a given n, the formula for I_n is given by*

$$I_n = h \sum_{k=0}^{n} \alpha_k f(x_k), \qquad \textit{where } \alpha_k = \alpha_{n-k}.$$

The order of error is estimated by

$$I - I_n = \begin{cases} O(h^{n+2} \|f^{(n+1)}\|_\infty) & \textit{if } n \textit{ is odd} \\ O(h^{n+3} \|f^{(n+2)}\|_\infty) & \textit{if } n \textit{ is even.} \end{cases}$$

Proof. Put $p(x) = f(a + hx)$. Then $p(k) = f(x_k)$ $(k = 0, 1, \ldots, n-1)$ and

$$I = \int_a^b f(x)dx = \int_0^n f(a + ht)hdt = h \int_0^n p(x)dx.$$

By Newton's formula, the polynomial that interpolates $p(k)$ $(k = 0, 1, \ldots, n-1)$ is given by

$$\begin{aligned} \varphi(x) &= \sum_{k=0}^{n} \frac{p(k)}{\prod_{j \neq k}(k-j)} \cdot \frac{W(x)}{x-k} \\ &= \sum_{k=0}^{n} f(x_k) n! \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{n-k} \cdot \frac{W(x)}{x-k}, \qquad \text{with } W(x) = \prod_{i=0}^{n} (x-i). \end{aligned}$$

Hence,

$$I_n = h \int_0^n \varphi(x)dx = h \sum_{k=0}^{n} \int_0^n \frac{W(x)\ dx}{x-k} n! \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{n-k} \cdot f(x_k) = h \sum_{k=0}^{n} \alpha_k f(x_k).$$

To show that $\alpha_k = \alpha_{n-k}$, we note that

$$\alpha_k = \frac{1}{n!} \int_0^n \frac{W(x)dx}{x-k} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{n-k}.$$

Hence, it follows that

$$\alpha_{n-k} = \frac{1}{n!} \int_0^n \frac{W(x)dx}{x-n+k} \begin{pmatrix} n \\ n-k \end{pmatrix} (-1)^{n-(n-k)}.$$

Changing the variable by $t = n - x$, and noting that $W(x) = W(n-t) = (-1)^{n+1} W(t)$, that ${}_nC_k =_n C_{n-k}$, and that $(-1)^k = (-1)^{-k}$, we have

$$\alpha_{n-k} = \frac{1}{n!} \int_0^n \frac{(-1)^{n+1} W(t)dt}{-t+k} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{-k} = \frac{1}{n!} \int_0^n \frac{W(t)dt}{t-k} \begin{pmatrix} n \\ k \end{pmatrix} (-1)^{n-k} = \alpha_k.$$

To evaluate the error, first we note that (F: a primitive of f)

$$\begin{aligned} I &= \int_a^b f(x)dx = F(b) - F(a) = F(a+nh) - F(a) \\ &= \sum_{l=0}^{\infty} \frac{F^{(l+1)}(a)}{(l+1)!}(nh)^{l+1} = \sum_{l=0}^{\infty} \frac{n^{l+1}}{(l+1)!} h^{l+1} f^{(l)}(a) =: \sum_{l=0}^{\infty} T_l(I) h^{l+1} f^{(l)}(a). \end{aligned}$$

On the other hand, recalculating I_n by using the following formula:

$$\varphi(x) = (1+\Delta)^x p(0) = \sum_{k=0}^{n} \binom{x}{k} \Delta^k p(0) = \sum_{k=0}^{n} \left[\binom{x}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} p(j) \right],$$

we have

$$I_n = h\int_0^n \varphi(x)dx = h\sum_{k=0}^{n}\left[\int_0^n \binom{x}{k} dx \cdot \sum_{j=0}^{k} (-1)^{k-j} p(j)\right].$$

Further, by substituting

$$p(j) = \sum_{l=0}^{\infty} \frac{j^l}{l!} p^{(l)}(0) = \sum_{l=0}^{\infty} \frac{j^l}{l!} h^l f^{(l)}(a),$$

we have

$$\begin{aligned} I_n &= h\sum_{k=0}^{n}\left[\int_0^n \binom{x}{k} dx \cdot \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \sum_{l=0}^{\infty} \frac{j^l}{l!} h^l f^{(l)}(a)\right] \\ &= \sum_{l=0}^{\infty}\left[\sum_{k=0}^{n}\int_0^n \binom{x}{k} dx \cdot \left(\sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} j^l\right) \frac{h^{l+1}}{l!} f^{(l)}(a)\right] \\ &= \sum_{l=0}^{\infty}\left[\sum_{k=0}^{n}\int_0^n \binom{x}{k} dx \cdot (\Delta^k z^l\big|_{z=0}) \frac{h^{l+1}}{l!} f^{(l)}(a)\right] \\ &= \sum_{l=0}^{\infty}\left[\int_0^n \left(\sum_{k=0}^{n}\int_0^n \binom{x}{k} (\Delta^k z^l\big|_{z=0}) dx\right) \frac{h^{l+1}}{l!} f^{(l)}(a)\right] \\ &=: \sum_{l=0}^{\infty} T_l(I_n) h^{l+1} f^{(l)}(a). \end{aligned}$$

In the above, we put

$$T_l(I_n) = \int_0^n G_l(x)dx, \qquad \text{where } G_l(x) = \frac{1}{l!}\sum_{k=0}^{n} \binom{x}{k} (\Delta^k z^l\big|_{z=0}).$$

For $l \leqq n$, noting that $\Delta^k z^l = 0$ (for $k > l$), we have

$$G_l(x) = \frac{1}{l!}\sum_{k=0}^{n} \binom{x}{k} (\Delta^k z^l|_{z=0}) = \frac{1}{l!}\sum_{k=0}^{l} \binom{x}{k} (\Delta^k z^l\big|_{z=0}) = \frac{x^l}{l!} \text{ (by Newton's formula).}$$

Hence, for $l \leqq n$,

$$T_l(I_n) = \int_0^n G_l(x)dx = \frac{1}{l!}\int_0^n x^l \, dx = \frac{n^{l+1}}{(l+1)!} = T_l(I).$$

Therefore,

$$\begin{aligned} I - I_n &= \sum_{l=0}^{\infty}(T_l(I) - T_l(I_n))h^{l+1}f^{(l)}(a) \\ &= \sum_{l=n+1}^{\infty}(T_l(I) - T_l(I_n))h^{l+1}f^{(l)}(a) = O(h^{n+2}\|f^{(n+1)}\|_\infty). \end{aligned}$$

Further,

$$\begin{aligned} G_{n+1}(x) &= \frac{1}{(n+1)!}\sum_{k=0}^{n}\binom{x}{k}(\Delta^k z^{n+1}\Big|_{z=0}) \\ &= \frac{1}{(n+1)!}\left[\sum_{k=0}^{n+1}\binom{x}{k}(\Delta^k z^{n+1}\Big|_{z=0}) - \binom{x}{n+1}(\Delta^{n+1}z^{n+1}\Big|_{z=0})\right] \\ &= \frac{x^{n+1}}{(n+1)!} - \frac{\Delta^{n+1}z^{n+1}\Big|_{z=0}}{(n+1)!}\binom{x}{n+1}. \end{aligned}$$

Hence, if $n = 2m$, the last-term in the above has the following property:

$$\binom{x}{n+1} = \binom{x}{2m+1} = -\binom{2m-x}{2m+1}.$$

Therefore this function is an odd function with respect to the axis $x = m$. Therefore, its integral over $[0, n] = [0, 2m]$ vanishes. We thus obtain $T_{n+1}(I_n) = T_{n+1}(I)$, which shows that the coefficient of the next higher terms of the error equals zero. Thus, $I - I_n = O(h^{n+3}\|f^{(n+2)}\|_\infty)$. (Q.E.D.)

Remark. The general estimation : $I - I_n = O(h^{n+2}\|f^{(n+1)}\|_\infty)$, can also be obtained by integration of the formula in Theorem II.1.4. However, in order to get a better estimation for even n, further detailed discussion is necessary.

Table. Figures of A, B_i (where $\alpha_i = AB_i$) and C (where $I - I_n \leqq Ch^{2[n/2]+3}\|f^{(2[n/2]+2)}\|_\infty$, for $1 \leqq n \leqq 8$)

n	A	B_0	B_1	B_2	B_3	B_4	C
1	1/2	1	1	-	-	-	-1/12
2	1/3	1	4	1	-	-	-1/90
3	3/8	1	3	3	1	-	-3/80
4	2/45	7	32	12	32	7	-8/945
5	5/288	19	75	50	50	75	-275/12096
6	1/140	41	216	27	272	27	-9/1400
7	7/17280	751	3577	1323	2989	2989	-8183/518400
8	4/14175	989	5888	-928	10496	-4540	-2368/467775

In practice, when the length of $[a, b]$ is large, this interval is equally divided into subintervals and the above integration method is applied for each subinterval. We summarise the most widely used formulae:

Corollary. (Compound rules)

(i) (Trapezoidal rule) *If $[a, b]$ is divided into N subintervals and the Newton-Cotes formula of n=1 is applied, we have*

$$I = h\left[\frac{1}{2}(f(a)+f(b)) + \sum_{k=1}^{N-1} f(a+kh)\right] + O(h^2\|f^{(2)}\|_\infty),$$

where $h = (b-a)/N$. (Note that the change in the order of error is due to the summation over $N = (b-a)/h$ subintervals).

(ii) (Compound Simpson's rule) *If $[a, b]$ is divided into N (N: even) subintervals and the Newton-Cotes formula of $n = 2$ is applied, we have*

$$I = \frac{h}{3}\left[(f(a)+f(b)) + 2\sum_{k=1}^{\frac{N}{2}-1} f(a+2kh) + 4\sum_{k=1}^{\frac{N}{2}} f(a+(2k-1)h)\right] + O(h^4\|f^{(4)}\|_\infty),$$

where $h = (b-a)/N$.

The Euler-Maclaurin summation formula gives a different insight into the numerical integration. Applying the Euler-Maclaurin summation formula for $g(x) = f(a+hx)$ with $h = (b-a)/n$, we have

$$\begin{aligned}\int_0^n f(a+hx)dx &= \sum_{k=0}^{n} f(a+kh) - \frac{f(a)+f(b)}{2} + \sum_{j=1}^{\infty}\frac{B_{2j}}{(2j)!}\cdot\left(\frac{d}{dx}\right)^{2j-1} f(a+hx)\Big|_{x=0}^{n} \\ &= \frac{f(a)+f(b)}{2} + \sum_{k=1}^{n-1} f(a+kh) + \sum_{j=1}^{\infty}\frac{B_{2j}}{(2j)!}\cdot f^{2j-1}(x)\Big|_{x=a}^{b} h^{2j-1}.\end{aligned}$$

Hence,

$$\int_a^b f(x)dx = h\int_0^n f(a+hx)dx = h\left[\frac{f(a)+f(b)}{2} + \sum_{k=1}^{n-1} f(a+kh)\right] + \sum_{j=1}^{\infty}\frac{B_{2j}}{(2j)!}\cdot f^{2j-1}(x)\Big|_{x=a}^{b} h^{2j}.$$

Note that the trapezoidal rule is given by

$$J_n = h\left[f(a) + 2\sum_{j=1}^{n-1} f(a+jh) + f(b)\right].$$

Therefore, the above formula shows an estimation of error of $I = \int_a^b f(x)dx$ by the trapezoidal rule,

$$I - J_n = \sum_{j=1}^{\infty}\left[\frac{B_{2j}}{(2j)!} f^{(2j-1)}(x)\Big|_{x=a}^{b}\right] h^{2j}.$$

This suggests that the trapezoidal rule gives high accuracy if the integrand is an periodical analytic function (the integration is taken over a period). It should also be noted that the trapezoidal rule gives quite high accuracy when applied to the rapidly decreasing function over an infinite interval.

Remark. Note that the right-hand side of the above equation does not necessarily converge but it gives an asymptotic expansion of the error. Also, compare this formula with the following one which can be derived from the proof of Theorem II.3.1,

$$I - I_1 = \sum_{l=2}^{\infty} (T_l(I) - T_l(I_1)) h^l f^{(l)}(a) = O(h^2 \|f^{(2)}\|_\infty).$$

If we divide $[a, b]$ into $n/2$, then the mesh equals $(b-a)/(n/2) = 2h$, and

$$I - J_{n/2} = \sum_{j=1}^{\infty} \left[\frac{B_{2j}}{(2j)!} f^{(2j-1)}(x) \Big|_{x=a}^{b} \right] (2h)^{2j} = \sum_{j=1}^{\infty} 2^{2j} \left[\frac{B_{2j}}{(2j)!} f^{(2j-1)}(x) \Big|_{x=a}^{b} \right] h^{2j}.$$

Hence,

$$4(I - J_n) - (I - J_{n/2}) = \sum_{j=2}^{\infty} (4 - 2^{2j}) \left[\frac{B_{2j}}{(2j)!} f^{(2j-1)}(x) \Big|_{x=a}^{b} \right] h^{2j}.$$

Therefore, $J_n^{(1)} = (4J_n - J_{n/2})/(4-1)$ improves the order of error from $O(h^2)$ to $O(h^4)$. In a similar manner, the recursion formula

$$J_n^{(k)} = \frac{4^k J_n^{(k-1)} - J_{n/2}^{(k-1)}}{4^k - 1}$$

gives an iterative approximation of I. This method is called the Romberg integration.

Concluding Remarks: Applications to Actuarial Mathematics

1. Formal relation of operators and formulae of interest rates

One may notice that the resemblance between the formal identity of operators:

$$e^{\partial} = 1 + \Delta = \left(1 + \frac{\Delta^{(m)}}{m}\right)^m,$$

and the formulae of the force of interest, the effective rate of interest, and the nominal rate of interest (compounded m times a year):

$$e^{\partial} = 1 + i = \left(1 + \frac{i^{(m)}}{m}\right)^m \quad \left[= \frac{F(t+1)}{F(t)}\right].$$

Here,

δ	:	Force of interest.
i	:	Effective rate of interest.
$i^{(m)}$	:	Nominal rate of interest(compounded m times a year).
$F(t)$	:	Amount of fund in investment at time t.

The above relation may be understood by the fact that $i, i^{(m)}$ and δ can be written as eigenvalues of the operators $\Delta, \Delta^{(m)}$ and ∂, with $F(t)$ being their eigenfunction in common.

$$\begin{aligned}
&(1+\Delta)F(t) = F(t+1) = (1+i)F(t);\\
&\left(1 + \frac{\Delta^{(m)}}{m}\right)F(t) = F\left(t + \frac{1}{m}\right) = \left(1 + \frac{i^{(m)}}{m}\right)F(t);\\
&\partial F(t) = F'(t) = \delta F(t).
\end{aligned}$$

2. Application of numerical differentiation to approximate the mortality force

Apply the numerical differentiation to approximate the force of mortality defined by:

$$\mu_x = -\frac{1}{l_x} \cdot \frac{dl_x}{dx}.$$

By using the central difference formula[1], we obtain

$$\mu_x = -\frac{\mu\delta \cdot l_x}{l_x} = \frac{l_{x-1} - l_{x+1}}{2l_x}.$$

[1] Note that μx on the right side denotes the force of mortality at age x, and μ in the middle denotes the mean difference operator, defined in II.1.2.

By applying the five points approximation, we obtain

$$\mu_x = -\frac{1}{l_x}\left(\mu\delta \cdot l_x - \frac{1}{6}\mu\delta^3 \cdot l_x\right) = \frac{8(l_{x-1} - l_{x+1}) - (l_{x-2} - l_{x+2})}{12 l_x}.$$

3. Application of the Euler-Maclaurin summation formula for continuous annuities

The Euler-Maclaurin summation formula is usually applied for the approximation of a defined integral in the following form:

$$\int_M^N f(x)dx = \sum_{k=M}^{N} f(k) - \frac{1}{2}(f(M) + f(N)) - \sum_{j\geq 1} \frac{B_{2j}}{(2j)!} f^{(2j-1)}(x)\Big|_{x=M}^{N}.$$

A typical application in actuarial mathematics is the estimation of the continuous annuity, as shown in the following:

$$\begin{aligned}
\overline{a}_x &= \frac{1}{D_x}\int_0^\infty D_{x+t}dt \\
&= \frac{1}{D_x}\left[\sum_{t=0}^{\infty} D_{x+t} - \frac{1}{2}D_x + \frac{1}{12}\frac{dD_x}{dx} + \text{higher-terms}\right] \\
&= \ddot{a}_x - \frac{1}{2} - \frac{1}{12}(\mu_x + \delta) + \text{etc.}
\end{aligned}$$

where $D_x = l_x \cdot v^x$. In deriving the above formula, we have used the following relation:

$$D'_x = \frac{d}{dx}(l_x v^x) = (-l_x\mu_x)v^x + l_x(-\delta v^x) = -D_x(\mu_x + \delta).$$

Taking the first two terms will result in the following approximation:

$$\overline{a}_x \simeq \ddot{a}_x - \frac{1}{2}.$$

Putting $\delta = 0$ in the continuous annuity $\overline{a}_x$ will give the life expectancy $\overset{\circ}{e}_x$. Therefore, replacing D_x by l_x in the above, we have

$$\overset{\circ}{e}_x = \frac{1}{l_x}\int_0^\infty l_{x+t}dt = \frac{1}{l_x}\sum_{t=0}^{\infty} l_{x+t} - \frac{1}{2} - \frac{1}{12}\mu_x + \text{etc.}$$

By taking the first two terms, we have

$$\overset{\circ}{e}_x \simeq E_x - \frac{1}{2}, \qquad \text{where } E_x = \frac{1}{l_x}\sum_{t=0}^{\infty} l_{x+t}.$$

4. Application of the Woolhouse summation formula for annuities payable more frequently than once a year

A typical application of the Woolhouse summation formula is to approximate the annuity of a unit amount divided into $\frac{1}{m}$ and payable m times a year. In usual actuarial notation, the present value may be written

$$\ddot{a}_x^{(m)} = \frac{1}{l_x}\sum_{t=0}^{\infty}\frac{1}{m}\cdot l_{x+\frac{t}{m}}\cdot v^{\frac{t}{m}} = \frac{1}{mD_x}\sum_{t=0}^{\infty}D_{x+\frac{t}{m}}.$$

Applying the Woolhouse summation formula, we have

$$\begin{aligned}&\frac{1}{mD_x}\sum_{t=0}^{\infty}D_{x+\frac{t}{m}}\\ =&\frac{1}{D_x}\left[\sum_{t=0}^{\infty}D_{x+t}-\frac{m-1}{2m}(D_x+D_\infty)-\sum_{j\geq 1}\frac{B_{2j}}{(2j)!}\frac{m^{2j}-1}{m^{2j}}\left(\frac{d}{dt}\right)^{2j-1}D_{x+t}\Big|_{t=0}^{\infty}\right].\end{aligned}$$

Therefore, taking the first derivative in the last summation term, we obtain the following approximation:

$$\ddot{a}_x^{(m)} \simeq \frac{1}{D_x}\sum_{t=0}^{\infty}D_{x+t}-\frac{m-1}{2m}-\frac{1}{12}\cdot\frac{m^2-1}{m^2}\cdot(\mu_x+\delta).$$

or

$$\ddot{a}_x^{(m)} \simeq \ddot{a}_x-\frac{m-1}{2m}-\frac{1}{12}\cdot\frac{m^2-1}{m^2}\cdot(\mu_x+\delta).$$

Note that $D_x=0$ (for $x>\omega$, ω: ultimate age). Verify that $\lim_{m\to\infty}\ddot{a}_x^{(m)}=\overline{a}_x$.

Remarks. (i) Since $D_x = l_x\cdot v^x = l_x\cdot e^{-\delta x}$ and l_x is bounded by $l_0>0$, the condition [$**$] stated in section 3.4 in Part I is met if $0<\delta<2\pi$. Note that $\delta=2\pi \iff i=534.49\cdots$, thus almost all actual cases (provided the interest rate is less than 53449%), the reminder term of the Euler-Maclaurin summation formula and that of the Woolhouse summation formula converge to zero.

(ii) Assuming the linearity of D_x between $x+t$ and $x+t+1$, i.e.,

$$(\dagger)\qquad D_{x+t+\frac{t}{m}} \simeq D_{x+t}+\frac{i}{m}(D_{x+t+1}-D_{x+t}),$$

can also lead to a simpler approximation. In fact,

$$\begin{aligned}
\ddot{a}_x^{(m)} &= \frac{1}{mD_x}\sum_{t=0}^{\infty}\sum_{i=0}^{m-1} D_{x+t+\frac{t}{m}} \\
&\simeq \frac{1}{mD_x}\sum_{t=0}^{\infty}\sum_{i=0}^{m-1} D_{x+t} + \frac{1}{m^2}\cdot\sum_{t=0}^{\infty}\frac{D_{x+t+1}-D_{x+t}}{D_x}\cdot\sum_{i=0}^{m-1} i \\
&= \frac{1}{D_x}\sum_{t=0}^{\infty} D_{x+t} + \frac{1}{m^2}\cdot\frac{D_\infty - D_x}{D_x}\cdot\frac{(m-1)m}{2} \\
&= \ddot{a}_x - \frac{m-1}{2m}.
\end{aligned}$$

(iii) In the above, consider the meaning of assumption (†). For instance in the case of mid year, $D_{x+\frac{1}{2}} = \dfrac{1}{2}(D_x + D_{x+1})$, i.e., $l_{x+\frac{1}{2}}v^{x+\frac{1}{2}} = \frac{1}{2}(l_x v^x + l_{x+1}v^{x+1})$. Further, without taking into account survival probability, this is further reduced to $v^{x+\frac{1}{2}} = \frac{1}{2}(v^x + v^{x+1})$.

References

Hairer, E. and Wanner, G. (1995). *Analysis by Its History*, UTM. Springer.

Bourbaki, N. (1961). *Fonctions d'une variable réelle (Théorie élémentaire)* (Deuxième édition) Hermann.

Whittaker, E. T. and Watson, G. N. (1927) *A course of modern analysis* (4th ed.) Cambridge Univ. Press.

Dieudonné, J. (1980). *Calcul infinitésimal* (Deuxième édition), Hermann, Paris.

Scheid, F. (1989). *Numerical Analysis* (2nd ed.), McGraw Hill.

Sugihara, M. and Murota, K. (1998). *Methods of Numerical Analysis* (in Japanese), Iwanami shoten publisher, Tokyo.

Iri, M. and Fujino, Y. (1985). *Common Practices in Numerical Analysis* (in Japanese), Kyoritsu shuppan, Tokyo.

Sakamoto, J. (1980, 1981). "On Woolhouse's Formula", *Journal of Society of Actuaries of Japan.* No. 33 and 34.

Fujita, T. (1992). *Graduation* (in Japanese), *Journal of Society of Actuaries of Japan. Suppl.* No. 134.

Neill, A. (1977). *Life Contingencies,* Heinemann, London.

Bibliographical Notes

1. Financial Indicators and Systems of Financing of Social Security Pensions

Based on the paper "A generalisation of the concept of the scaled premium" presented at the 11th International Conference of Social Security Actuaries and Statisticians held in Athens from 19 to 21 June 1995. The original paper was published in the proceedings of the Conference *Social Security Financing : Issues and Perspectives* (International Social Security Association, Geneva, 1996. Translations into French, Spanish and German are also available.)

This paper owes its origin to discussions with Mr. Yoshihiro Yumiba. The author would also like to thank Mr. Warren McGillivray, Mr. Michael Cichon, Mr. Jean-Paul Picard and Ms. Anne Drouin for reading the draft and making helpful comments.

2. On the Scaled Premium System

Survey article on this subject. I thank Mr. Subramaniam Iyer for his comments and discussions.

3. A Relation Between the PAYG Contribution Rate and the Entry Age Normal Premium Rate

It is known that when the real interest rate is more than the growth rate of the working population, a funding scheme is more favourable than a PAYG scheme, and vice versa. Despite the familiarity of this assertion, the known proofs are somehow obscure. In this note, I have tried to clarify this statement and the underlying assumptions.

4. Natural Cubic Spline Interpolation

In this technical note, I have tried to show explicitly the algorithm to calculate coefficients. Compared, for instance, with the Sprague formulae, this method requires more calculation but is more flexible in many technical respects.

The Excel VBA programme attached to this note was written by Mr. Florian Ljer.

5. Sprague Interpolation

The publication *The Methods and Materials of Demography*, by Shryock, H.S. and Siegel, J.S. (U.S. Department of Commerce, 1973) provides coefficients of various interpolation methods including the Sprague interpolation formulae (Appendix C).

The original motivation was to verify these results by deriving them from the basic principles. As a result, alternative sets of coefficients were proposed for the end-points.

6. An Interpolation of Abridged Mortality Rates

The crucial assumption of the method explained in this paper is to apply a polynomial interpolation for the abridged force of mortality. This method was first established by Mr. Atsuo Koike for the purpose of constructing the life tables of Japan. Due to the improvement of the statistics, this method is no longer used in Japan. However, for his statistical work including this interpolation method, he was awarded the Ouchi prize in 1991, which is awarded to statisticians who have made remarkable contributions to the progress in statistics.

7. The Concept of Stable Population

The purpose of this paper is to give self-contained proofs for this well known result, developed by A. J. Lotka and his colleagues.

The first part (continuous approach) is based on McKendrick's model explained in Hoppensteadt, F. C. : *Mathematical methods of population biology*, 1982. Cambridge Univ. Press. (pp.44-45).

I am grateful to Mr. Masanori Akatsuka for pointing out some errors in the earlier draft and providing valuable comments, many of which have been incorporated into this final version. Ms. Elisabeth Davi provided technical assistance in the drafting of this paper.

8. Theory of Lorenz Curves and its Applications to Income Distribution Analysis

The original intention was to calculate the Lorenz curves and Gini coefficients for as many statistical distributions as possible. This was achieved in Part II.

The general theory in Part I was intended to improve the already known results. As a consequence, the method of Lorenz curves and Gini coefficients has been extended to a quite general class of distributions. The main reference is found in the bibliography in Atkinson, A.B.: *Social Justice and Public Policy*, 1983. Harvester Wheatsheaf (p. 35).

I am grateful to Mr. Hiroshi Yamabana for stimulating discussions and for his advice on the probability theory exploited in this paper. Mr. Akatsuka provided useful comments. Ms. Elisabeth Davi and Mr. Nicolas Serrire provided technical assistance in the drafting of this paper.

9. Notes on Lognormal and Multivariate Normal Distributions

This paper was motivated by the following remark made by Mr. Michael Cichon: Consider a defined benefit pension scheme. A simple way to estimate the newly awarded pension is to assume that all retired workers have the same average salary and average length of service and to apply the pension formula to these average figures. However, there are some situations in which an estimation based on averages may give rise to a significant deviation. For instance, it is essential to consider explicitly the distributions in order to estimate the following effects:

(i) The effect of the minimum pension and its increase. The percentage of pensioners who fall into the minimum pensioners category.

(ii) The effect of the distribution of length of services, in case the benefit rate is not fully proportional to the period of contributions.

(iii) The effect of the correlation between the salary and contribution years.

This paper attempts to provide a simple, but plausible, two-dimensional distribution which enables us to estimate these effects.

Ms. Elisabeth Davi provided technical assistance in the drafting of this paper. Ms. Antje Kesseler wrote the Excel VBA programme attached to this note.

10. On Some Issues in Actuarial Mathematics

This note was built up through an attempt to clarify some issues which I encountered in my study of actuarial mathematics. The applications to numerical analysis (Part II) were obtained as a byproduct of this clarification. As noted in the preface, this paper can be used as the mathematical reference for other papers, in particular, papers 4-6. I am grateful to Mr. Yoshihiro Yumiba and Mr. Hiroshi Yamabana for their comments.

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