15 BASIC INDEX NUMBER THEORY

Introduction

> The answer to the question what is the Mean of a given set of magnitudes cannot in general be found, unless there is given also the object for the sake of which a mean value is required. There are as many kinds of average as there are purposes; and we may almost say in the matter of prices as many purposes as writers. Hence much vain controversy between persons who are literally at cross purposes. (Edgeworth (1888, p. 347)).

15.1 The number of physically distinct goods and unique types of services that consumers can purchase is in the millions. On the business or production side of the economy, there are even more commodities that are actively traded. This is because firms not only produce commodities for final consumption, but they also produce exports and intermediate commodities that are demanded by other producers. Firms collectively also use millions of imported goods and services, thousands of different types of labour services and hundreds of thousands of specific types of capital. If we further distinguish physical commodities by their geographical location or by the season or time of day that they are produced or consumed, then there are billions of commodities that are traded within each year in any advanced economy. For many purposes, it is necessary to summarize this vast amount of price and quantity information into a much smaller set of numbers. The question that this chapter addresses is: how exactly should the microeconomic information involving possibly millions of prices and quantities be aggregated into a smaller number of price and quantity variables? This is the basic problem of index numbers.

15.2 It is possible to pose the index number problem in the context of microeconomic theory; i.e., given that we wish to implement some economic model based on producer or consumer theory, what is the “best” method for constructing a set of aggregates for the model? When constructing aggregate prices or quantities, however, other points of view (that do not rely on economics) are possible. Some of these alternative points of view are considered in this chapter and the next. Economic approaches are pursued in Chapters 17 and 18.

15.3 The index number problem can be framed as the problem of decomposing the value of a well-defined set of transactions in a period of time into an aggregate price term times an aggregate quantity term. It turns out that this approach to the index number problem does not lead to any useful solutions. So, in paragraphs 15.7 to 15.17, the problem of decomposing a value ratio pertaining to two periods of time into a component that measures the overall change in prices between the two periods (this is the price index) times a term that measures the overall change in quantities between the two periods (this is the quantity index) is considered. The simplest price index is a fixed basket type index; i.e., fixed amounts of the *n* quantities in the value aggregate are chosen and then the values of this fixed basket of quantities at the prices of period 0 and at the prices of period 1 are calculated. The fixed basket price index is simply the ratio of these two values where the prices vary but the quantities are held fixed. Two natural choices for the fixed basket are the quantities transacted in the base period, period 0, or the quantities transacted in the current period, period 1. These two choices lead to the Laspeyres (1871) and Paasche (1874) price indices, respectively.

15.4 Unfortunately, the Paasche and Laspeyres measures of aggregate price change can differ, sometimes substantially. Thus in paragraphs 15.18 to 15.32, taking an average of these two indices to come up with a single measure of price change is considered. In paragraphs 15.18 to 15.23, it is argued that the "best" average to take is the geometric mean, which is Irving Fisher's (1922) ideal price index. In paragraphs 15.24 to 15.32, instead of averaging the Paasche and Laspeyres measures of price change, taking an average of the two baskets is considered. This fixed basket approach to index number theory leads to a price index advocated by Correa Moylan Walsh (1901; 1921a). Other fixed basket approaches are, however, also possible. Instead of choosing the basket of period 0 or 1 (or an average of these two baskets), it is possible to choose a basket that pertains to an entirely different period, say period *b*. In fact, it is typical statistical agency practice to pick a basket that pertains to an entire year (or even two years) of transactions in a year prior to period 0, which is usually a month. Indices of this type, where the weight reference period differs from the price reference period, were originally proposed by Joseph Lowe (1823), and indices of this type are studied in paragraphs 15.24 to 15.53. Such indices are also evaluated from the axiomatic perspective in Chapter 16 and from the economic perspective in Chapter 17.[1]

[1] Although indices of this type do not appear in Chapter 19, where most of the index number formulae exhibited in Chapters 15–18 are illustrated using an artificial data set, indices where the weight reference period differs from the price reference period are illustrated numerically in Chapter 22, in which the problem of seasonal commodities is discussed.

15.5 In paragraphs 15.65 to 15.75, another approach to the determination of the *functional form* or the *formula* for the price index is considered. This approach is attributable to the French economist Divisia (1926) and is based on the assumption that price and quantity data are available as continuous functions of time. The theory of differentiation is used in order to decompose the rate of change of a continuous time value aggregate into two components that reflect aggregate price and quantity change. Although the approach of Divisia offers some insights,[2] it does not offer much guidance to statistical agencies in terms of leading to a definite choice of index number formula.

15.6 In paragraphs 15.76 to 15.97, the advantages and disadvantages of using a *fixed base* period in the bilateral index number comparison are considered versus always comparing the current period with the previous period, which is called the *chain system*. In the chain system, a *link* is an index number comparison of one period with the previous period. These links are multiplied together in order to make comparisons over many periods.

The decomposition of value aggregates into price and quantity components

The decomposition of value aggregates and the product test

15.7 A *price index* is a measure or function which summarizes the *change* in the prices of many commodities from one situation 0 (a time period or place) to another situation 1. More specifically, for most practical purposes, a price index can be regarded as a weighted mean of the change in the relative prices of the commodities under consideration in the two situations. To determine a price index, it is necessary to know:

- which commodities or items to include in the index;
- how to determine the item prices;
- which transactions that involve these items to include in the index;
- how to determine the weights and from which sources these weights should be drawn;
- what formula or type of mean should be used to average the selected item relative prices.

All the above questions regarding the definition of a price index, except the last, can be

[2] In particular, it can be used to justify the chain system of index numbers (discussed in paragraphs 15.86 to 15.97).

answered by appealing to the definition of the *value aggregate* to which the price index refers. A value aggregate V for a given collection of items and transactions is computed as:

$$V = \sum_{i=1}^{n} p_i q_i \qquad (15.1)$$

where p_i represents the price of the ith item in national currency units, q_i represents the corresponding quantity transacted in the time period under consideration and the subscript i identifies the ith elementary item in the group of n items that make up the chosen value aggregate V. Included in this definition of a value aggregate is the specification of the group of included commodities (which items to include) and of the economic agents engaging in transactions involving those commodities (which transactions to include), as well as principles of the valuation and time of recording that motivate the behaviour of the economic agents undertaking the transactions (determination of prices). The included elementary items, their valuation (the p_i), the eligibility of the transactions and the item weights (the q_i) are all within the domain of definition of the value aggregate. The precise determination of the p_i and q_i is discussed in more detail elsewhere in this manual, in particular in Chapter 5.[3]

15.8 The value aggregate V defined by equation (15.1) refers to a certain set of transactions pertaining to a single (unspecified) time period. Now the same value aggregate for two places or time periods, periods 0 and 1, is considered. For the sake of convenience, period 0 is called the *base period* and period 1 is called the *current period* and it is assumed that observations on the base period price and quantity vectors, $p^0 \equiv [p_1^0,\ldots,p_n^0]$ and $q^0 \equiv [q_1^0,\ldots,q_n^0]$ respectively, have been collected.[4] The value aggregates in the base and current periods are defined in the obvious way as:

$$V^0 \equiv \sum_{i=1}^{n} p_i^0 q_i^0 ; \quad V^1 \equiv \sum_{i=1}^{n} p_i^1 q_i^1 \qquad (15.2)$$

[3] Ralph Turvey has noted that some values may be difficult to decompose into unambiguous price and quantity components. Examples of difficult-to-decompose values are bank charges, gambling expenditures and life insurance payments.

[4] Note that it is assumed that there are no new or disappearing commodities in the value aggregates. Approaches to the “new goods problem” and the problem of accounting for quality change are discussed in Chapters 7, 8 and 21.

In the previous paragraph, a price index was defined as a function or measure which summarizes the change in the prices of the n commodities in the value aggregate from situation 0 to situation 1. In this paragraph, a *price index* $P(p^0,p^1,q^0,q^1)$ along with the corresponding *quantity index* (or *volume index*) $Q(p^0,p^1,q^0,q^1)$ is defined to be two functions of the $4n$ variables p^0,p^1,q^0,q^1 (these variables describe the prices and quantities pertaining to the value aggregate for periods 0 and 1) where these two functions satisfy the following equation:[5]

$$V^1/V^0 = P(p^0,p^1,q^0,q^1)\,Q(p^0,p^1,q^0,q^1) \qquad (15.3)$$

If there is only one item in the value aggregate, then the price index P should collapse down to the single price ratio, p_1^1/p_1^0, and the quantity index Q should collapse down to the single quantity ratio, q_1^1/q_1^0. In the case of many items, the price index P is to be interpreted as some sort of weighted average of the individual price ratios, $p_1^1/p_1^0,\ldots, p_n^1/p_n^0$.

15.9 Thus the first approach to index number theory can be regarded as the problem of decomposing the change in a value aggregate, V^1/V^0, into the product of a part that is attributable to *price change*, $P(p^0,p^1,q^0,q^1)$, and a part that is attributable to *quantity change*, $Q(p^0,p^1,q^0,q^1)$. This approach to the determination of the price index is the approach that is taken in the national accounts, where a price index is used to deflate a value ratio in order to obtain an estimate of quantity change. Thus, in this approach to index number theory, the primary use for the price index is as a *deflator*. Note that once the functional form for the price index $P(p^0,p^1,q^0,q^1)$ is known, then the corresponding quantity or volume index $Q(p^0,p^1,q^0,q^1)$ is completely determined by P; i.e., rearranging equation (15.3):

$$Q(p^0,p^1,q^0,q^1) = \left(V^1/V^0\right)/P(p^0,p^1,q^0,q^1) \qquad (15.4)$$

Conversely, if the functional form for the quantity index $Q(p^0,p^1,q^0,q^1)$ is known, then the corresponding price index $P(p^0,p^1,q^0,q^1)$ is completely determined by Q. Thus using this deflation approach to index number theory, separate theories for the determination of the price and quantity indices are not required: if either P or Q is determined, then the other function is implicitly determined by the product test equation (15.4).

[5] The first person to suggest that the price and quantity indices should be jointly determined in order to satisfy equation (15.3) was Fisher (1911, p. 418). Frisch (1930, p. 399) called equation (15.3) the *product test*.

15.10 In the next section, two concrete choices for the price index $P(p^0,p^1,q^0,q^1)$ are considered and the corresponding quantity indices $Q(p^0,p^1,q^0,q^1)$ that result from using equation (15.4) are also calculated. These are the two choices used most frequently by national income accountants.

The Laspeyres and Paasche indices

15.11 One of the simplest approaches to the determination of the price index formula was described in great detail by Lowe (1823). His approach to measuring the price change between periods 0 and 1 was to specify an approximate *representative commodity basket*,[6] which is a quantity vector $q \equiv [q_1,\ldots,q_n]$ that is representative of purchases made during the two periods under consideration, and then calculate the level of prices in period 1 relative to period 0 as the ratio of the period 1 cost of the basket, $\sum_{i=1}^{n} p_i^1 q_i$, to the period 0 cost of the basket, $\sum_{i=1}^{n} p_i^0 q_i$. This *fixed basket approach* to the determination of the price index leaves open the question as to how exactly is the fixed basket vector q to be chosen.

15.12 As time passed, economists and price statisticians demanded a little more precision with respect to the specification of the basket vector q. There are two natural choices for the reference basket: the base period commodity vector q^0 or the current period commodity vector q^1. These two choices lead to the Laspeyres (1871) price index[7] P_L defined by equation (15.5) and the Paasche (1874) price index[8] P_P defined by equation (15.6):[9]

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[6] Lowe (1823, Appendix, p. 95) suggested that the commodity basket vector q should be updated every five years. Lowe indices are studied in more detail in paragraphs 15.24 to 15.53.

[7] This index was actually introduced and justified by Drobisch (1871a, p. 147) slightly earlier than Laspeyres. Laspeyres (1871, p. 305) in fact explicitly acknowledged that Drobisch showed him the way forward. However, the contributions of Drobisch have been forgotten for the most part by later writers because Drobisch aggressively pushed for the ratio of two unit values as being the “best” index number formula. While this formula has some excellent properties where all the n commodities being compared have the same unit of measurement, it is useless when, say, both goods and services are in the index basket.

[8] Drobisch (1871b, p. 424) also appears to have been the first to define explicitly and justify the Paasche price index formula, but he rejected this formula in favour of his preferred formula, the ratio of unit values, and so again he did not gain any credit for his early suggestion of the Paasche formula.

[9] Note that $P_L(p^0,p^1,q^0,q^1)$ does not actually depend on q^1 and $P_P(p^0,p^1,q^0,q^1)$ does not actually depend on q^0. It does no harm to include these vectors, however, and the notation indicates that the reader is in the realm of

$$P_L(p^0, p^1, q^0, q^1) \equiv \frac{\sum_{i=1}^{n} p_i^1 q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0} \tag{15.5}$$

$$P_P(p^0, p^1, q^0, q^1) \equiv \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^1} \tag{15.6}$$

15.13 The formulae (15.5) and (15.6) can be rewritten in an alternative manner that is more useful for statistical agencies. Define the period *t* expenditure share on commodity *i* as follows:

$$s_i^t \equiv p_i^t q_i^t \Big/ \sum_{j=1}^{n} p_j^t q_j^t \quad \text{for } i = 1,\ldots,n \quad \text{and} \quad t = 0,1 \tag{15.7}$$

Then the Laspeyres index (15.5) can be rewritten as follows:[10]

$$\begin{aligned} P_L(p^0, p^1, q^0, q^1) &= \sum_{i=1}^{n} p_i^1 q_i^0 \Big/ \sum_{i=1}^{n} p_j^0 q_j^0 \\ &= \sum_{i=1}^{n} (p_i^1 / p_i^0) p_i^0 q_i^0 \Big/ \sum_{j=1}^{n} p_j^0 q_j^0 \\ &= \sum_{i=1}^{n} (p_i^1 / p_i^0) s_i^0 \end{aligned} \tag{15.8}$$

using definitions (15.7). The Laspeyres price index P_L can thus be written as an arithmetic average of the *n* price ratios, p_i^1/p_i^0, weighted by base period expenditure shares. The Laspeyres formula (until very recently) has been widely used as the intellectual base for

bilateral index number theory; i.e., the prices and quantities for a value aggregate pertaining to two periods are being compared.

[10] This method of rewriting the Laspeyres index (or any fixed basket index) as a share weighted arithmetic average of price ratios is attributable to Fisher (1897, p. 517) (1911, p. 397) (1922, p. 51) and Walsh (1901, p. 506; 1921a, p. 92).

consumer price indices (CPIs) around the world. To implement it, a statistical agency needs only to collect information on expenditure shares s_n^0 for the index domain of definition for the base period 0, and then collect information on item *prices* alone on an ongoing basis. *Thus the Laspeyres CPI can be produced on a timely basis without having quantity information for the current period.*

15.14 The Paasche index can also be written in expenditure share and price ratio form as follows:[11]

$$
\begin{aligned}
P_P(p^0,p^1,q^0,q^1) &= \frac{1}{\left\{\sum_{i=1}^{n} p_i^0 q_i^1 \Big/ \sum_{j=1}^{n} p_j^1 q_j^1\right\}} \\
&= \frac{1}{\left\{\sum_{i=1}^{n} \left(p_i^0/p_i^1\right) p_i^1 q_i^1 \Big/ \sum_{j=1}^{n} p_j^1 q_j^1\right\}} \\
&= \frac{1}{\left\{\sum_{i=1}^{n} \left(p_i^1/p_i^0\right)^{-1} s_i^1\right\}} \\
&= \left\{\sum_{i=1}^{n} \left(p_i^1/p_i^0\right)^{-1} s_i^1\right\}^{-1} \qquad (15.9)
\end{aligned}
$$

using definitions (15.7). The Paasche price index P_P can thus be written as a *harmonic* average of the *n* item price ratios, p_i^1/p_i^0, weighted by period 1 (current period) expenditure shares.[12] The lack of information on current period quantities prevents statistical agencies from producing Paasche indices on a timely basis.

15.15 The quantity index that corresponds to the Laspeyres price index using the product test in equation (15.3) is the Paasche quantity index; i.e., if *P* in equation (15.4) is replaced by

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[11] This method of rewriting the Paasche index (or any fixed basket index) as a share weighted harmonic average of the price ratios is attributable to Walsh (1901, p. 511; 1921a, p. 93) and Fisher (1911, p. 397-398).

[12] Note that the derivation in the formula (15.9) shows how harmonic averages arise in index number theory in a very natural way.

P_L defined by equation (15.5), then the following quantity index is obtained:

$$Q_P(p^0, p^1, q^0, q^1) \equiv \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{i=1}^{n} p_i^1 q_i^0} \qquad (15.10)$$

Note that Q_P is the value of the period 1 quantity vector valued at the period 1 prices, $\sum_{i=1}^{n} p_i^1 q_i^1$, divided by the (hypothetical) value of the period 0 quantity vector valued at the period 1 prices, $\sum_{i=1}^{n} p_i^1 q_i^0$. Thus the period 0 and 1 quantity vectors are valued at the same set of prices, the current period prices, p^1.

15.16 The quantity index that corresponds to the Paasche price index using the product test (15.3) is the Laspeyres quantity index; i.e., if P in equation (15.4) is replaced by P_P defined by equation (15.6), then the following quantity index is obtained:

$$Q_L(p^0, p^1, q^0, q^1) \equiv \frac{\sum_{i=1}^{n} p_i^0 q_i^1}{\sum_{i=1}^{n} p_i^0 q_i^0} \qquad (15.11)$$

Note that Q_L is the (hypothetical) value of the period 1 quantity vector valued at the period 0 prices, $\sum_{i=1}^{n} p_i^0 q_i^1$, divided by the value of the period 0 quantity vector valued at the period 0 prices, $\sum_{i=1}^{n} p_i^0 q_i^0$. Thus the period 0 and 1 quantity vectors are valued at the same set of prices, the base period prices, p^0.

15.17 The problem with the Laspeyres and Paasche index number formulae is that, although they are equally plausible, in general they will give different answers. For most purposes, it is not satisfactory for the statistical agency to provide two answers to the question[13]: What is the "best" overall summary measure of price change for the value aggregate over the two periods in question? In the following section, we consider how "best" averages of these two estimates of price change can be constructed. Before doing so, we ask: What is the "normal" relationship between the Paasche and Laspeyres indices? Under "normal" economic conditions when the price ratios pertaining to the two situations under consideration are negatively correlated with the corresponding quantity ratios, it can be shown that the Laspeyres price index will be larger than the corresponding Paasche index.[14] A precise statement of this result is presented in Appendix 15.1.[15] The divergence between P_L and P_P suggests that if a *single estimate* for the price change between the two periods is required, then some sort of evenly weighted average of the Laspeyres and Paasche indices should be taken as the final estimate of price change between periods 0 and 1. As mentioned above, this strategy will be pursued in the following section. It should, however, be kept in mind that statistical agencies will not usually have information on current expenditure weights, hence averages of Paasche and Laspeyres indices can be produced only on a delayed basis (perhaps using national accounts information) or not at all.

Symmetric averages of fixed basket price indices

The Fisher index as an average of the Paasche and Laspeyres indices

15.18 As mentioned above, since the Paasche and Laspeyres price indices are equally

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[13] In principle, instead of averaging the Paasche and Laspeyres indices, the statistical agency could think of providing both (the Paasche index on a delayed basis). This suggestion would lead to a matrix of price comparisons between every pair of periods instead of a time series of comparisons. Walsh (1901, p. 425) noted this possibility: "In fact, if we use such direct comparisons at all, we ought to use all possible ones."

[14] Peter Hill (1993, p. 383) summarized this inequality as follows:

> It can be shown that relationship (13) [i.e., that P_L is greater than P_P] holds whenever the price and quantity relatives (weighted by values) are negatively correlated. Such negative correlation is to be expected for price takers who react to changes in relative prices by substituting goods and services that have become relatively less expensive for those that have become relatively more expensive. In the vast majority of situations covered by index numbers, the price and quantity relatives turn out to be negatively correlated so that Laspeyres indices tend systematically to record greater increases than Paasche with the gap between them tending to widen with time.

[15] There is another way to see why P_P will often be less than P_L. If the period 0 expenditure shares s_i^0 are exactly equal to the corresponding period 1 expenditure shares s_i^1, then by Schlömilch's (1858) Inequality (see Hardy, Littlewood and Polyá (1934, p. 26)), it can be shown that a weighted harmonic mean of n numbers is equal to or less than the corresponding arithmetic mean of the n numbers and the inequality is strict if the n numbers are not all equal. If expenditure shares are approximately constant across periods, then it follows that P_P will usually be less than P_L under these conditions (see paragraphs 15.70 to 15.84).

plausible but often give different estimates of the amount of aggregate price change between periods 0 and 1, it is useful to consider taking an evenly weighted average of these fixed basket price indices as a single estimator of price change between the two periods. Examples of such *symmetric averages*[16] are the arithmetic mean, which leads to the Drobisch (1871b, p. 425), Sidgwick (1883, p. 68) and Bowley (1901, p. 227)[17] index, $P_D \equiv (1/2)P_L + (1/2)P_P$, and the geometric mean, which leads to the Fisher (1922)[18] ideal index, P_F, defined as

$$P_F(p^0, p^1, q^0, q^1) \equiv \{P_L(p^0, p^1, q^0, q^1)\, P_P(p^0, p^1, q^0, q^1)\}^{1/2} \qquad (15.12)$$

At this point, the fixed basket approach to index number theory is transformed into the *test approach* to index number theory; i.e., in order to determine which of these fixed basket indices or which averages of them might be "best", desirable *criteria* or *tests* or *properties* are needed for the price index. This topic will be pursued in more detail in the next chapter, but an introduction to the test approach is provided in the present section because a test is used to determine which average of the Paasche and Laspeyres indices might be "best".

15.19 What is the "best" symmetric average of P_L and P_P to use as a point estimate for the theoretical cost of living index? It is very desirable for a price index formula that depends on the price and quantity vectors pertaining to the two periods under consideration to satisfy the *time reversal test.*[19] An index number formula $P(p^0,p^1,q^0,q^1)$ satisfies this test if

$$P(p^1, p^0, q^1, q^0) = 1/\, P(p^0, p^1, q^0, q^1) \qquad (15.13)$$

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[16] For a discussion of the properties of symmetric averages, see Diewert (1993c). Formally, an average m(a,b) of two numbers a and b is symmetric if m(a,b) = m(b,a). In other words, the numbers a and b are treated in the same manner in the average. An example of a nonsymmetric average of a and b is (1/4)a + (3/4)b. In general, Walsh (1901, p. 105) argued for a symmetric treatment if the two periods (or countries) under consideration were to be given equal importance.

[17] Walsh (1901, p. 99) also suggested the arithmetic mean index P_D (see Diewert (1993a, p. 36) for additional references to the early history of index number theory).

[18] Bowley (1899, p.641) appears to have been the first to suggest the use of the geometric mean index P_F. Walsh (1901, p. 428-429) also suggested this index while commenting on the big differences between the Laspeyres and Paasche indices in one of his numerical examples: "The figures in columns (2) [Laspeyres] and (3) [Paasche] are, singly, extravagant and absurd. But there is order in their extravagance; for the nearness of their means to the more truthful results shows that they straddle the true course, the one varying on the one side about as the other does on the other."

[19] See Diewert (1992a, p. 218) for early references to this test. If we want the price index to have the same property as a single price ratio, then it is important to satisfy the time reversal test. However, other points of view are possible. For example, we may want to use our price index for compensation purposes, in which case satisfaction of the time reversal test may not be so important.

i.e., if the period 0 and period 1 price and quantity data are interchanged, and then the index number formula is evaluated, then this new index $P(p^1,p^0,q^1,q^0)$ is equal to the reciprocal of the original index $P(p^0,p^1,q^0,q^1)$. This is a property that is satisfied by a single price ratio, and it seems desirable that the measure of aggregate price change should also satisfy this property so that it does not matter which period is chosen as the base period. Put another way, the index number comparison between any two points of time should not depend on the choice of which period we regard as the base period: if the other period is chosen as the base period, then the new index number should simply equal the reciprocal of the original index. It should be noted that the Laspeyres and Paasche price indices do not satisfy this time reversal property.

15.20 Having defined what it means for a price index P to satisfy the time reversal test, then it is possible to establish the following result.[20] The Fisher ideal price index defined by equation (15.12) is the *only* index that is a homogeneous[21] symmetric average of the Laspeyres and Paasche price indices, P_L and P_P, and satisfies the time reversal test (15.13). The Fisher ideal price index thus emerges as perhaps the "best" evenly weighted average of the Paasche and Laspeyres price indices.

15.21 It is interesting to note that this *symmetric basket approach* to index number theory dates back to one of the early pioneers of index number theory, Arthur L. Bowley, as the following quotations indicate:

> If [the Paasche index] and [the Laspeyres index] lie close together there is no further difficulty; if they differ by much they may be regarded as inferior and superior limits of the index number, which may be estimated as their arithmetic mean … as a first approximation (Bowley (1901, p. 227)).
>
> When estimating the factor necessary for the correction of a change found in money wages to obtain the change in real wages, statisticians have not been content to follow Method II only [to calculate a Laspeyres price index], but have worked the problem backwards [to calculate a Paasche price index] as well as forwards. … They have then taken the arithmetic, geometric or harmonic mean of the two numbers so found (Bowley (1919, p. 348)).[22]

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[20] See Diewert (1997, p. 138))

[21] An average or mean of two numbers *a* and *b*, *m(a,b),* is *homogeneous* if when both numbers *a* and *b* are multiplied by a positive number λ, then the mean is also multiplied by λ; i.e., m satisfies the following property: $m(\lambda a,\lambda b) = \lambda m(a,b)$.

[22] Fisher (1911, p. 417-418; 1922) also considered the arithmetic, geometric and harmonic averages of the

15.22 The quantity index that corresponds to the Fisher price index using the product test (15.3) is the Fisher quantity index; i.e., if P in equation (15.4) is replaced by P_F defined by equation (15.12), the following quantity index is obtained:

$$Q_F(p^0,p^1,q^0,q^1) \equiv \{Q_L(p^0,p^1,q^0,q^1)Q_P(p^0,p^1,q^0,q^1)\}^{1/2} \quad (15.14)$$

Thus the Fisher quantity index is equal to the square root of the product of the Laspeyres and Paasche quantity indices. It should also be noted that $Q_F(p^0,p^1,q^0,q^1) = P_F(q^0,q^1,p^0,p^1)$; i.e., if the role of prices and quantities is interchanged in the Fisher price index formula, then the Fisher quantity index is obtained. [23]

15.23 Rather than take a symmetric average of the two basic fixed basket price indices pertaining to two situations, P_L and P_P, it is also possible to return to Lowe's basic formulation and choose the basket vector q to be a symmetric average of the base and current period basket vectors, q^0 and q^1. This approach to index number theory is pursued in the following section.

The Walsh index and the theory of the "pure" price index

15.24 Price statisticians tend to be very comfortable with a concept of the price index that is based on pricing out a constant "representative" basket of commodities, $q \equiv (q_1,q_2,\ldots,q_n)$, at the prices of periods 0 and 1, $p^0 \equiv (p_1^0,p_2^0,\ldots,p_n^0)$ and $p^1 \equiv (p_1^1,p_2^1,\ldots,p_n^1)$ respectively. Price statisticians refer to this type of index as a *fixed basket index* or a *pure price index*[24] and it corresponds to Sir George H. Knibbs's (1924, p. 43) *unequivocal price index*.[25] Since Lowe

Paasche and Laspeyres indices.

[23] Fisher (1922, p. 72) said that P and Q satisfied the *factor reversal test* if $Q(p^0,p^1,q^0,q^1) = P(q^0,q^1,p^0,p^1)$ and P and Q satisfied the product test (15.3) as well.

[24] See section 7 in Diewert (2001).

[25] "Suppose however that, for each commodity, Q′ = Q, then the fraction, ∑(P′Q) / ∑(PQ), viz., the ratio of aggregate value for the second unit-period to the aggregate value for the first unit-period is no longer merely a ratio of totals, it also shows unequivocally the effect of the change in price. Thus it is an unequivocal price index for the quantitatively unchanged complex of commodities, A, B, C, etc.

It is obvious that if the quantities were different on the two occasions, and if at the same time the prices had been unchanged, the preceding formula would become ∑(PQ′) / ∑(PQ). It would still be the ratio of the aggregate value for the second unit-period to the aggregate value for the first unit period. But it would be also more than this. It would show in a generalized way the ratio of the quantities on the two occasions. Thus it is an unequivocal quantity index for the complex of commodities, unchanged as to price and differing only as to quantity.

(1823) was the first person to describe systematically this type of index, it is referred to as a Lowe index. Thus the general functional form for the *Lowe price index* is

$$P_{Lo}(p^0, p^1, q) \equiv \sum_{i=1}^{n} p_i^1 q_i \Big/ \sum_{i=1}^{n} p_i^0 q_i = \sum_{i=1}^{n} s_i (p_i^1 / p_i^0) \qquad (15.15)$$

where the (hypothetical) *hybrid expenditure shares* s_i[26] corresponding to the quantity weights vector q are defined by:

$$s_i \equiv p_i^0 q_i \Big/ \sum_{j=1}^{n} p_j^0 q_j \quad \text{for } i = 1,2,...,n \qquad (15.16)$$

15.25 The main reason why price statisticians might prefer a member of the family of Lowe or fixed basket price indices defined by equation (15.15) is that the fixed basket concept is easy to explain to the public. Note that the Laspeyres and Paasche indices are special cases of the pure price concept if we choose $q = q^0$ (which leads to the Laspeyres index) or if we choose $q = q^1$ (which leads to the Paasche index).[27] The practical problem of picking q remains to be resolved, and that is the problem that will be addressed in this section.

15.26 It should be noted that Walsh (1901, p. 105; 1921a) also saw the price index number problem in the above framework:

> Commodities are to be weighted according to their importance, or their full values. But the problem of axiometry always involves at least two periods. There is a first period, and there is a second period which is compared with it. Price variations have taken place between the two, and these are to be averaged to get the amount of their variation as a whole. But the weights of the commodities at the second period are apt to be different from their weights at the first period. Which weights, then, are the right ones—those of the first period? Or those of the second? Or should there be a combination of the two sets? There is no reason for preferring either the first or the second. Then the combination of both would seem to be the proper answer. And this combination itself involves an averaging of the weights

Let it be noted that the mere algebraic form of these expressions shows at once the logic of the problem of finding these two indices is identical" (Knibbs (1924, p. 43–44)).

[26] Note that Fisher (1922, p. 53) used the terminology "weighted by a hybrid value", while Walsh (1932, p. 657) used the term "hybrid weights".

[27] Note that the *i*th share defined by equation (15.16) in this case is the hybrid share $s_i \equiv p_i^0 q_i^1 / \sum_{i=1}^{n} p_i^0 q_i^1$, which uses the prices of period 0 and the quantities of period 1.

of the two periods (Walsh (1921a, p. 90)).

Walsh's suggestion will be followed and thus the *i*th quantity weight, q_i, is restricted to be an average or *mean* of the base period quantity q_i^0 and the current period quantity for commodity *i* q_i^1, say $m(q_i^0,q_i^1)$, for $i = 1,2,\ldots, n$.[28] Under this assumption, the Lowe price index (15.15) becomes:

$$P_{Lo}(p^0,p^1,q^0,q^1) = \frac{\sum_{i=1}^{n} p_i^1 m(q_i^0,q_i^1)}{\sum_{j=1}^{n} p_j^0 m(q_j^0,q_j^1)}. \qquad (15.17)$$

15.27 In order to determine the functional form for the mean function *m*, it is necessary to impose some *tests* or *axioms* on the pure price index defined by equation (15.17). As above, we ask that P_{Lo} satisfy the *time reversal test* (15.13). Under this hypothesis, it is immediately obvious that the mean function m must be a *symmetric mean*[29]; i.e., *m* must satisfy the following property: $m(a,b) = m(b,a)$ for all $a > 0$ and $b > 0$. This assumption still does not pin down the functional form for the pure price index defined by equation (15.17). For example, the function *m(a,b)* could be the *arithmetic mean*, $(1/2)a + (1/2)b$, in which case equation (15.17) reduces to the *Marshall* (1887) and *Edgeworth* (1925) *price index* P_{ME}, which was the pure price index preferred by Knibbs (1924, p. 56):

$$P_{ME}(p^0,p^1,q^0,q^1) \equiv \frac{\sum_{i=1}^{n} p_i^1 \left\{\left(q_i^0+q_i^1\right)/2\right\}}{\sum_{j=1}^{n} p_j^0 \left\{\left(q_j^0+q_j^1\right)/2\right\}} \qquad (15.18)$$

15.28 On the other hand, the function *m(a,b)* could be the *geometric mean*, $(ab)^{1/2}$, in which case equation (15.17) reduces to the *Walsh* (1901, p. 398; 1921a, p. 97) *price index*, P_W:[30]

[28] Note that we have chosen the mean function $m(q_i^0,q_i^1)$ to be the same for each item *i*. We assume that *m(a,b)* has the following two properties: *m(a,b)* is a positive and continuous function, defined for all positive numbers *a* and *b* and $m(a,a) = a$ for all $a > 0$.

[29] For more on symmetric means, see Diewert (1993c, p. 361).

[30] Walsh (1921a, p. 103) endorsed P_W as being the best index number formula: "We have seen reason to believe formula 6 better than formula 7. Perhaps formula 9 is the best of the rest, but between it and Nos. 6 and 8 it would be difficult to decide with assurance". His formula 6 is P_W defined by equation (15.19) and his 9 is the

$$P_W(p^0, p^1, q^0, q^1) \equiv \frac{\sum_{i=1}^{n} p_i^1 \sqrt{q_i^0 q_i^1}}{\sum_{j=1}^{n} p_j^0 \sqrt{q_j^0 q_j^1}} \tag{15.19}$$

15.29 There are many other possibilities for the mean function m, including the mean of order r, $[(1/2)a^r + (1/2)b^r]^{1/r}$ for $r \neq 0$. Obviously, in order to completely determine the functional form for the pure price index P_{Lo}, it is necessary to impose at least one additional test or axiom on $P_{Lo}(p^0,p^1,q^0,q^1)$.

15.30 There is a potential problem with the use of the Edgeworth-Marshall price index (15.18) that has been noticed in the context of using the formula to make international comparisons of prices. If the price levels of a very large country are compared to the price levels of a small country using formula (15.18), then the quantity vector of the large country may totally overwhelm the influence of the quantity vector corresponding to the small country.[31] In technical terms, the Edgeworth-Marshall formula is not homogeneous of degree 0 in the components of both q^0 and q^1. To prevent this problem from occurring in the use of the pure price index $P_K(p^0,p^1,q^0,q^1)$ defined by equation (15.17), it is asked that P_{Lo} satisfy the following *invariance to proportional changes in current quantities test:*[32]

$$P_{Lo}(p^0, p^1, q^0, \lambda q^1) = P_{Lo}(p^0, p^1, q^0, q^1) \quad \text{for all } p^0, p^1, q^0, q^1 \quad \text{and all } \lambda > 0 \tag{15.20}$$

The two tests, the time reversal test (15.13) and the invariance test (15.20), make it possible to determine the precise functional form for the pure price index P_{Lo} defined by formula (15.17): the pure price index P_K must be the Walsh index P_W defined by formula (15.19).[33]

15.31 In order to be of practical use by statistical agencies, an index number formula must

Fisher ideal defined by equation (15.12). The *Walsh quantity index*, $Q_W(p^0,p^1,q^0,q^1)$ is defined as $P_W(q^0,q^1,p^0,p^1)$; i.e., the role of prices and quantities in definition (15.19) is interchanged. If the Walsh quantity index is used to deflate the value ratio, an implicit price index is obtained, which is Walsh's formula 8.

[31] This is not likely to be a severe problem in the time series context, however, where the change in quantity vectors going from one period to the next is small.

[32] This is the terminology used by Diewert (1992a, p. 216); Vogt (1980) was the first to propose this test.

[33] See section 7 in Diewert (2001).

be able to be expressed as a function of the base period expenditure shares, s_i^0, the current period expenditure shares, s_i^1, and the n price ratios, p_i^1/p_i^0. The Walsh price index defined by the formula (15.19) can be rewritten in the following format:

$$P_W(p^0, p^1, q^0, q^1) \equiv \frac{\sum_{i=1}^{n} p_i^1 \sqrt{q_i^0 q_i^1}}{\sum_{j=1}^{n} p_j^0 \sqrt{q_j^0 q_j^1}}$$

$$= \frac{\sum_{i=1}^{n} \left(p_i^1 \Big/ \sqrt{p_i^0 p_i^1} \right) \sqrt{s_i^0 s_i^1}}{\sum_{j=1}^{n} \left(p_j^0 \Big/ \sqrt{p_j^0 p_j^1} \right) \sqrt{s_j^0 s_j^1}}$$

$$= \frac{\sum_{i=1}^{n} \sqrt{s_i^0 s_i^1} \sqrt{p_i^1 / p_i^0}}{\sum_{j=1}^{n} \sqrt{s_j^0 s_j^1} \sqrt{p_j^0 / p_j^1}} \qquad (15.21)$$

15.32 The approach taken to index number theory in this section was to consider averages of various fixed basket type price indices. The first approach was to take an even-handed average of the two primary fixed basket indices: the Laspeyres and Paasche price indices. These two primary indices are based on pricing out the baskets that pertain to the two periods (or locations) under consideration. Taking an average of them led to the Fisher ideal price index P_F defined by equation (15.12). The second approach was to average the basket quantity weights and then price out this average basket at the prices pertaining to the two situations under consideration. This approach led to the Walsh price index, P_W, defined by equation (15.19). Both of these indices can be written as a function of the base period expenditure shares, s_i^0, the current period expenditure shares, s_i^1, and the n price ratios, p_i^1/p_i^0. Assuming that the statistical agency has information on these three sets of variables, which index should be used? Experience with normal time series data has shown that these two indices will not differ substantially and thus it is a matter of indifference which of these indices is used in practice.[34] Both of these indices are examples of *superlative indices*, which are defined in Chapter 17. Note, however, that both of these indices treat the data pertaining to the two situations in a *symmetric* manner. Hill[35] commented on superlative price indices and the importance of a symmetric treatment of the data as follows:

> Thus economic theory suggests that, in general, a symmetric index that assigns equal weight to the two situations being compared is to be preferred to either the Laspeyres or Paasche indices on their own. The precise choice of superlative index—whether Fisher, Törnqvist or other superlative index—may be of only secondary importance as all the symmetric indices are likely to approximate each other, and the underlying theoretic index fairly closely, at least when the index number spread between the Laspeyres and Paasche is not very great (Hill (1993, p. 384)).

Annual weights and monthly price indices

The Lowe index with monthly prices and annual base year quantities

15.33 It is now necessary to discuss a major practical problem with the above theory of basket type indices. Up to now, it has been assumed that the quantity vector $q \equiv (q_1,q_2,\ldots,q_n)$ that appeared in the definition of the Lowe index, $P_{Lo}(p^0,p^1,q)$ defined by equation (15.15), is either the base period quantity vector q^0 or the current period quantity vector q^1 or an average of these two quantity vectors. In fact, in terms of actual statistical agency practice, the quantity vector q is usually taken to be an annual quantity vector that refers to a *base year*, say b, that is prior to the base period for the prices, period 0. Typically, a statistical agency will produce a consumer price index at a monthly or quarterly frequency, but for the sake of argument a monthly frequency will be assumed in what follows. Thus a typical price index will have the form $P_{Lo}(p^0,p^t,q^b)$, where p^0 is the price vector pertaining to the base period month for prices, month 0, p^t is the price vector pertaining to the current period month for prices, say month t, and q^b is a reference basket quantity vector that refers to the base year b, which is equal to or prior to month 0.[36] Note that this Lowe index $P_{Lo}(p^0,p^t,q^b)$ is *not* a true Laspeyres index (because the annual quantity vector q^b is not equal to the monthly quantity vector q^0 in general).[37]

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[34] Diewert (1978, pp. 887-889) showed that these two indices will approximate each other to the second order around an equal price and quantity point. Thus for normal time series data where prices and quantities do not change much going from the base period to the current period, the indices will approximate each other quite closely.

[35] See also Hill (1988).

[36] Month 0 is called the price reference period and year b is called the weight reference period.

[37] Triplett (1981, p. 12) defined the Lowe index, calling it a Laspeyres index, and calling the index that has the weight reference period equal to the price reference period, a pure Laspeyres index. Balk (1980c, p. 69), however, asserted that although the Lowe index is of the fixed base type; it is not a Laspeyres price index.

15.34 The question is: why do statistical agencies *not* pick the reference quantity vector q in the Lowe formula to be the monthly quantity vector q^0 that pertains to transactions in month 0 (so that the index would reduce to an ordinary Laspeyres price index)? There are two main reasons why this is not done:

- Most economies are subject to seasonal fluctuations, and so picking the quantity vector of month 0 as the reference quantity vector for all months of the year would not be representative of transactions made throughout the year.
- Monthly household quantity or expenditure weights are usually collected by the statistical agency using a household expenditure survey with a relatively small sample. Hence the resulting weights are usually subject to very large sampling errors and so standard practice is to average these monthly expenditure or quantity weights over an entire year (or in some cases, over several years), in an attempt to reduce these sampling errors.

The index number problems that are caused by seasonal monthly weights are studied in more detail in Chapter 22. For now, it can be argued that the use of annual weights in a monthly index number formula is simply a method for dealing with the seasonality problem.[38]

15.35 One problem with using annual weights corresponding to a perhaps distant year in the context of a monthly consumer price index must be noted at this point: if there are systematic (but divergent) trends in commodity prices and households increase their purchases of commodities that decline (relatively) in price and reduce their purchases of commodities that increase (relatively) in price, then the use of distant quantity weights will tend to lead to an upward bias in this Lowe index compared to one that used more current weights, as will be shown below. This observation suggests that statistical agencies should strive to get up-to-date weights on an ongoing basis.

Triplett also noted the hybrid share representation for the Lowe index defined by equations (15.15) and (15.16). Triplett noted that the ratio of two Lowe indices using the same quantity weights was also a Lowe index. Baldwin (1990, p. 255) called the Lowe index an *annual basket index*.

[38] In fact, the use of the Lowe index $P_{Lo}(p^0,p^t,q^b)$ in the context of seasonal commodities corresponds to Bean and Stine's (1924, p. 31) Type A index number formula. Bean and Stine made three additional suggestions for price indices in the context of seasonal commodities. Their contributions are evaluated in Chapter 22.

15.36 It is useful to explain how the annual quantity vector q^b could be obtained from monthly expenditures on each commodity during the chosen base year b. Let the month m expenditure of the reference population in the base year b for commodity i be $v_i^{b,m}$ and let the corresponding price and quantity be $p_i^{b,m}$ and $q_i^{b,m}$ respectively. Of course, value, price and quantity for each commodity are related by the following equations:

$$v_i^{b,m} = p_i^{b,m} q_i^{b,m} \quad \text{where} \quad i = 1,\ldots,n \quad \text{and} \quad m = 1,\ldots,12 \qquad (15.22)$$

For each commodity i, the annual total, q_i^{b} can be obtained by price deflating monthly values and summing over months in the base year b as follows:

$$q_i^b = \sum_{m=1}^{12} \frac{v_i^{b,m}}{p_i^{b,m}} = \sum_{m=1}^{12} q_i^{b,m}; \qquad i = 1,\ldots,n \qquad (15.23)$$

where equation (15.22) was used to derive the second equation in (15.23). In practice, the above equations will be evaluated using aggregate expenditures over closely related commodities and the price $p_i^{\mathrm{b,m}}$ will be the month m price index for this elementary commodity group i in year b relative to the first month of year b.

15.37 For some purposes, it is also useful to have annual prices by commodity to match up with the annual quantities defined by equation (15.23). Following national income accounting conventions, a reasonable[39] price p_i^b to match up with the annual quantity $q_i^{\mathbf{b}}$ is the value of total consumption of commodity i in year b divided by q_i^b.
Thus we have:

[39] These annual commodity prices are essentially unit value prices. Under conditions of high inflation, the annual prices defined by equation (15.24) may no longer be "reasonable" or representative of prices during the entire base year because the expenditures in the final months of the high inflation year will be somewhat artificially blown up by general inflation. Under these conditions, the annual prices and annual commodity expenditure shares should be interpreted with caution. For more on dealing with situations where there is high inflation within a year, see Hill (1996).

$$p_i^b \equiv \sum_{m=1}^{12} v_i^{b,m} / q_i^b \qquad i = 1,...,n$$

$$= \frac{\sum_{m=1}^{12} v_i^{b,m}}{\sum_{m=1}^{12} v_i^{b,m} / p_i^{b,m}} \qquad \text{using (15.23)}$$

$$= \left[\sum_{m=1}^{12} s_i^{b,m} (p_i^{b,m})^{-1} \right]^{-1} \qquad (15.24)$$

where the share of annual expenditure on commodity *i* in month m of the base year is

$$s_i^{b,m} \equiv \frac{v_i^{b,m}}{\sum_{k=1}^{12} v_i^{b,k}}; \qquad i = 1,...,n \qquad (15.25)$$

Thus the annual base year price for commodity *i*, p_i^b, turns out to be a monthly expenditure weighted *harmonic mean* of the monthly prices for commodity *i* in the base year, $p_i^{b,1}$, $p_i^{b,2}$,…, $p_i^{b,12}$.

Using the annual commodity prices for the base year defined by equation (15.24), a vector of these prices can be defined as $p^b \equiv [p_1^b,\ldots,p_n^b]$. Using this definition, the Lowe index $P_{Lo}(p^0,p^t,q^b)$ can be expressed as a ratio of two Laspeyres indices, where the price vector p^b plays the role of base period prices in each of the two Laspeyres indices:

$$P_{Lo}(p^0, p^t, q^b) = \frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b} = \frac{\sum_{i=1}^{n} p_i^t q_i^b / \sum_{i=1}^{n} p_i^b q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b / \sum_{i=1}^{n} p_i^b q_i^b}$$

$$= \frac{\sum_{i=1}^{n} s_i^b (p_i^t / p_i^b)}{\sum_{i=1}^{n} s_i^b (p_i^0 / p_i^b)}$$

$$= P_L(p^b, p^t, q^b) / P_L(p^b, p^0, q^b) \qquad (15.26)$$

where the Laspeyres formula P_L was defined by equation (15.5). Thus the above equation shows that the Lowe monthly price index comparing the prices of month 0 to those of month *t* using the quantities of base year *b* as weights, $P_{Lo}(p^0,p^t,q^b)$, is equal to the Laspeyres index that compares the prices of month *t* to those of year *b*, $P_L(p^b,p^t,q^b)$, divided by the Laspeyres index that compares the prices of month 0 to those of year *b*, $P_L(p^b,p^0,q^b)$. Note that the Laspeyres index in the numerator can be calculated if the base year commodity expenditure shares, s_i^b, are known along with the price ratios that compare the prices of commodity *i* in month t, p_i^t, with the corresponding annual average prices in the base year b, p_i^b. The Laspeyres index in the denominator can be calculated if the base year commodity expenditure shares, s_i^b, are known along with the price ratios that compare the prices of commodity *i* in month 0, p_i^0, with the corresponding annual average prices in the base year *b*, p_i^b.

15.39 There is another convenient formula for evaluating the Lowe index, $P_{Lo}(p^0,p^t,q^b)$, and that is to use the hybrid weights formula (15.15). In the present context, the formula becomes:

$$P_{Lo}(p^0, p^t, q^b) \equiv \frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b} = \frac{\sum_{i=1}^{n} \left(p_i^t / p_i^0 \right) p_i^0 q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b} = \sum_{i=1}^{n} \left(\frac{p_i^t}{p_i^0} \right) s_i^{0b} \qquad (15.27)$$

where the hybrid weights s_i^{0b} using the prices of month 0 and the quantities of year *b* are defined by

$$s_i^{0b} \equiv \frac{p_i^0 q_i^b}{\sum_{j=1}^{n} p_j^0 q_j^b}; \qquad i = 1,...,n$$

$$= \frac{p_i^b q_i^b (p_i^0 / p_i^b)}{\sum_{j=1}^{n} \left[p_j^b q_j^b (p_j^0 / p_j^b) \right]}. \qquad (15.28)$$

The second equation in (15.28) shows how the base year expenditures, $p_i^b q_i^b$, can be multiplied by the commodity price indices, p_i^0/p_i^b, in order to calculate the hybrid shares.

15.40 There is one additional formula for the Lowe index, $P_{Lo}(p^0,p^t,q^b)$, that will be exhibited. Note that the Laspeyres decomposition of the Lowe index defined by the third term in equation (15.26) involves the long-term price relatives, p_i^t/p_i^b, which compare the prices in month t, p_i^t, with the possibly distant base year prices, p_i^b, and that the hybrid share decomposition of the Lowe index defined by the third term in equation (15.27) involves the long-term monthly price relatives, p_i^t/p_i^0, which compare the prices in month t, p_i^t, with the base month prices, p_i^0. Both of these formulae are unsatisfactory in practice because of sample attrition: each month, a substantial fraction of commodities disappears from the marketplace. Thus it is useful to have a formula for updating the previous month's price index using just month-over-month price relatives. In other words, long-term price relatives disappear at too fast a rate to make it viable, in practice, to base an index number formula on their use. The Lowe index for month t+1, $P_{Lo}(p^0,p^{t+1},q^b)$, can be written in terms of the Lowe index for month t, $P_{Lo}(p^0,p^t,q^b)$, and an updating factor as follows:

$$P_{Lo}(p^0, p^{t+1}, q^b) \equiv \frac{\sum_{i=1}^{n} p_i^{t+1} q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b} = \left[\frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b}\right]\left[\frac{\sum_{i=1}^{n} p_i^{t+1} q_i^b}{\sum_{i=1}^{n} p_i^t q_i^b}\right]$$

$$= P_{Lo}(p^0, p^t, q^b)\left[\frac{\sum_{i=1}^{n} p_i^{t+1} q_i^b}{\sum_{i=1}^{n} p_i^t q_i^b}\right]$$

$$= P_{Lo}(p^0, p^t, q^b)\left[\frac{\sum_{i=1}^{n} \left(\frac{p_i^{t+1}}{p_i^t}\right) p_i^t q_i^b}{\sum_{i=1}^{n} p_i^t q_i^b}\right]$$

$$= P_{Lo}(p^0, p^t, q^b)\left[\sum_{i=1}^{n} \left(\frac{p_i^{t+1}}{p_i^t}\right) s_i^{tb}\right] \qquad (15.29)$$

where the hybrid weights s_i^{tb} are defined by:

$$s_i^{tb} \equiv \frac{p_i^t q_i^b}{\sum_{j=1}^{n} p_j^t q_j^b}; \qquad i = 1,\dots,n \qquad (15.30)$$

Thus the required updating factor, going from month t to month t+1, is the chain link index $\sum_{i=1}^{n} s_i^{tb}(p_i^{t+1}/p_i^t)$, which uses the hybrid share weights s_i^{tb} corresponding to month t and base year b.

15.41 The Lowe index $P_{Lo}(p^0, p^t, q^b)$ can be regarded as an approximation to the ordinary Laspeyres index, $P_L(p^0, p^t, q^0)$, that compares the prices of the base month 0, p^0, to those of month t, p^t, using the quantity vectors of month 0, q^0, as weights. It turns out that there is a relatively simple formula that relates these two indices. In order to explain this formula, it is first necessary to make a few definitions. Define the *i*th price relative between month 0 and month as

$$r_i \equiv p_i^t / p_i^0; \qquad i = 1,...,n \tag{15.31}$$

The ordinary Laspeyres price index, going from month 0 to *t*, can be defined in terms of these price relatives as follows:

$$P_L(p^0, p^t, q^0) \equiv \frac{\sum_{i=1}^{n} p_i^t q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0} = \frac{\sum_{i=1}^{n} \left(\frac{p_i^t}{p_i^0}\right) p_i^0 q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0}$$
$$= \sum_{i=1}^{n} \left(\frac{p_i^t}{p_i^0}\right) s_i^0 = \sum_{i=1}^{n} s_i^0 r_i \equiv r^* \tag{15.32}$$

where the month 0 expenditure shares s_i^0 are defined as follows:

$$s_i^0 \equiv \frac{p_i^0 q_i^0}{\sum_{j=1}^{n} p_j^0 q_j^0}; \qquad i = 1,...,n \tag{15.33}$$

15.42 Define the *i*th quantity relative t_i as the ratio of the quantity of commodity *i* used in the base year *b*, q_i^b, to the quantity used in month 0, q_i^0, as follows:

$$t_i \equiv q_i^b / q_i^0; \qquad i = 1,...,n \tag{15.34}$$

The Laspeyres quantity index, $Q_L(q^0, q^b, p^0)$, that compares quantities in year *b*, q^b, to the corresponding quantities in month 0, q^0, using the prices of month 0, p^0, as weights can be defined as a weighted average of the quantity ratios t_i as follows:

$$Q_L(q^0,q^b,p^0) \equiv \frac{\sum_{i=1}^{n} p_i^0 q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^0} = \frac{\sum_{i=1}^{n}\left(\frac{q_i^b}{q_i^0}\right) p_i^0 q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0} = \sum_{i=1}^{n}\left(\frac{q_i^b}{q_i^0}\right) s_i^0$$

$$= \sum_{i=1}^{n} s_i^0 t_i \qquad \text{using definition (15.34)}$$

$$\equiv t^* \qquad (15.35)$$

15.43 Using formula (A15.2.4) in Appendix 15.2 to this chapter, the relationship between the Lowe index $P_{Lo}(p^0,p^t,q^b)$ that uses the quantities of year b as weights to compare the prices of month t to month 0, and the corresponding ordinary Laspeyres index $P_L(p^0,p^t,q^0)$ that uses the quantities of month 0 as weights is the following one:

$$P_{Lo}(p^0,p^t,q^b) \equiv \frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b}$$

$$= P_L(p^0,p^t,q^0) + \frac{\sum_{i=1}^{n}(r_i - r^*)(t_i - t^*)s_i^0}{Q_L(q^0,q^b,p^0)} \qquad (15.36)$$

Thus the Lowe price index using the quantities of year b as weights, $P_{Lo}(p^0,p^t,q^b)$, is equal to the usual Laspeyres index using the quantities of month 0 as weights, $P_L(p^0,p^t,q^0)$, plus a covariance term $\sum_{i=1}^{n}(r_i - r^*)(t_i - t^*)s_i^0$ between the price relatives $r_i \equiv p_i^t/p_i^0$ and the quantity relatives $t_i \equiv q_i^b/q_i^0$, divided by the Laspeyres quantity index $Q_L(q^0, q^b, p^0)$ between month 0 and base year b.

15.44 Formula (15.36) shows that the Lowe price index will coincide with the Laspeyres price index if the covariance or correlation between the month 0 to t price relatives $r_i \equiv p_i^t/p_i^0$ and the month 0 to year b quantity relatives $t_i \equiv q_i^b/q_i^0$ is zero. Note that this covariance will be zero under three different sets of conditions:

- if the month t prices are proportional to the month 0 prices so that all $r_i = r^*$;
- if the base year b quantities are proportional to the month 0 quantities so that all $t_i = t^*$;
- if the distribution of the relative prices r_i is independent of the distribution of the relative quantities t_i.

The first two conditions are unlikely to hold empirically, but the third is possible, at least approximately, if consumers do not systematically change their purchasing habits in response to changes in relative prices.

15.45 If this covariance in formula (15.36) is negative, then the Lowe index will be less than the Laspeyres index. Finally, if the covariance is positive, then the Lowe index will be greater than the Laspeyres index. Although the sign and magnitude of the covariance term, $\sum_{i=1}^{n}(r_i - r^*)(t_i - t^*)s_i^0$, is ultimately an empirical matter, it is possible to make some reasonable conjectures about its likely sign. If the base year b precedes the price reference month 0 and there are long-term trends in prices, then it is likely that this covariance is positive and hence that the Lowe index will exceed the corresponding Laspeyres price index;[40] i.e.,

$$P_{Lo}(p^0, p^t, q^b) > P_L(p^0, p^t, q^0) \qquad (15.37)$$

To see why the covariance is likely to be positive, suppose that there is a long-term upward trend in the price of commodity i so that $r_i - r^* \equiv (p_i^t/p_i^0) - r^*$ is positive. With normal consumer substitution responses[41], q_i^t/q_i^0 less an average quantity change of this type is likely to be negative, or, upon taking reciprocals, q_i^0/q_i^t less an average quantity change of this

[40] For this relationship to hold, it is also necessary to assume that households have normal substitution effects in response to these long-term trends in prices; i.e., if a commodity increases (relatively) in price, its consumption will decline (relatively) and if a commodity decreases relatively in price, its consumption will increase relatively.

[41] Walsh (1901, pp. 281-282) was well aware of consumer substitution effects, as can be seen in the following comment which noted the basic problem with a fixed basket index that uses the quantity weights of a single period: "The argument made by the arithmetic averagist supposes that we buy the same quantities of every class at both periods in spite of the variation in their prices, which we rarely, if ever, do. As a rough proposition, we – a community – generally spend more on articles that have risen in price and get less of them, and spend less on articles that have fallen in price and get more of them."

(reciprocal) type is likely to be positive. But if the long-term upward trend in prices has persisted back to the base year b, then $t_i - t^* \equiv (q_i^b/q_i^0) - t^*$ is also likely to be positive. Hence, the covariance will be positive under these circumstances. Moreover, the more distant is the base year b from the base month 0, the bigger the residuals $t_i - t^*$ are likely to be and the bigger will be the positive covariance. Similarly, the more distant is the current period month t from the base period month 0, the bigger the residuals $r_i - r^*$ are likely to be and the bigger will be the positive covariance. Thus, under the assumptions that there are long-term trends in prices and normal consumer substitution responses, the Lowe index will normally be greater than the corresponding Laspeyres index.

15.46 Define the Paasche index between months 0 and t as follows:

$$P_P(p^0, p^t, q^t) \equiv \frac{\sum_{i=1}^{n} p_i^t q_i^t}{\sum_{i=1}^{n} p_i^0 q_i^t} \qquad (15.38)$$

As discussed in paragraphs 15.18 to 15.23, a reasonable target index to measure the price change going from month 0 to t is some sort of symmetric average of the Paasche index $P_P(p^0,p^t,q^t)$, defined by formula (15.38), and the corresponding Laspeyres index, $P_L(p^0,p^t,q^0)$, defined by formula (15.32). Adapting equation (A15.1.5) in Appendix 15.1, the relationship between the Paasche and Laspeyres indices can be written as follows:

$$P_P(p^0, p^t, q^t) = P_L(p^0, p^t, q^0) + \frac{\sum_{i=1}^{n} (r_i - r^*)(u_i - u^*) s_i^0}{Q_L(q^0, q^t, p^0)} \qquad (15.39)$$

where the price relatives $r_i \equiv p_i^t/p_i^0$ are defined by equation (15.31) and their share-weighted average r^* by equation (15.32) and the u_i, u^* and Q_L are defined as follows:

$$u_i \equiv q_i^t / q_i^0; \qquad i = 1,\ldots,n \qquad (15.40)$$

$$u^* \equiv \sum_{i=1}^{n} s_i^0 u_i = Q_L(q^0, q^t, p^0) \qquad (15.41)$$

and the month 0 expenditure shares s_i^0 are defined by the identity (15.33). Thus u^* is equal to the Laspeyres quantity index between months 0 and t. This means that the Paasche price index that uses the quantities of month t as weights, $P_P(p^0,p^t,q^t)$, is equal to the usual Laspeyres index using the quantities of month 0 as weights, $P_L(p^0,p^t,q^0)$, plus a covariance term $\sum_{i=1}^{n}(r_i - r^*)(u_i - u^*)s_i^0$ between the price relatives $r_i \equiv p_i^t/p_i^0$ and the quantity relatives $u_i \equiv q_i^t/q_i^0$, divided by the Laspeyres quantity index $Q_L(q^0,q^t,p^0)$ between month 0 and month t.

15.47 Although the sign and magnitude of the covariance term, $\sum_{i=1}^{n}(r_i - r^*)(u_i - u^*)s_i^0$, is again an empirical matter, it is possible to make a reasonable conjecture about its likely sign. If there are long-term trends in prices and consumers respond normally to price changes in their purchases, then it is likely that this covariance is negative and hence the Paasche index will be less than the corresponding Laspeyres price index; i.e.,

$$P_P(p^0, p^t, q^t) < P_L(p^0, p^t, q^0) \qquad (15.42)$$

To see why this covariance is likely to be negative, suppose that there is a long-term upward trend in the price of commodity i[42] so that $r_i - r^* \equiv (p_i^t/p_i^0) - r^*$ is positive. With normal consumer substitution responses, q_i^t/q_i^0 less an average quantity change of this type is likely to be negative. Hence $u_i - u^* \equiv (q_i^t/q_i^0) - u^*$ is likely to be negative. Thus, the covariance will be negative under these circumstances. Moreover, the more distant is the base month 0 from the current month t, the bigger in magnitude the residuals $u_i - u^*$ are likely to be and the bigger in magnitude will be the negative covariance.[43] Similarly, the more distant is the current period month t from the base period month 0, the bigger the residuals $r_i - r^*$ will probably be and the bigger in magnitude will be the covariance. Thus under the assumptions that there are long-term trends in prices and normal consumer substitution responses, the Laspeyres index will be greater than the corresponding Paasche index, with the divergence likely to grow as month t becomes more distant from month 0.

[42] The reader can carry through the argument if there is a long-term relative decline in the price of the *i*th commodity. The argument required to obtain a negative covariance requires that there be some differences in the long-term trends in prices; i.e., if all prices grow (or fall) at the same rate, there will be price proportionality and the covariance will be zero.

[43] However, $Q_L = u^*$ may also be growing in magnitude, so the net effect on the divergence between P_L and P_P is ambiguous.

15.48 Putting the arguments in the three previous paragraphs together, it can be seen that under the assumptions that there are long-term trends in prices and normal consumer substitution responses, the Lowe price index between months 0 and *t* will exceed the corresponding Laspeyres price index, which in turn will exceed the corresponding Paasche price index; i.e., under these hypotheses,

$$P_{Lo}(p^0, p^t, q^b) > P_L(p^0, p^t, q^0) > P_P(p^0, p^t, q^t) \qquad (15.43)$$

Thus, if the long-run target price index is an average of the Laspeyres and Paasche indices, it can be seen that the Laspeyres index will have an upward bias relative to this target index and the Paasche index will have a downward bias. In addition, if the base year *b* is prior to the price reference month, month 0, then the Lowe index will also have an upward bias relative to the Laspeyres index and hence also to the target index.

The Lowe index and mid-year indices

15.49 The discussion in the previous paragraph assumed that the base year *b* for quantities preceded the base month for prices, month 0. If the current period month *t* is quite distant from the base month 0, however, then it is possible to think of the base year *b* as referring to a year that lies between months 0 and *t*. If the year **b** does fall between months 0 and *t*, then the Lowe index becomes a *mid-year index*.[44] It turns out that the Lowe mid-year index no longer has the upward biases indicated by the inequalities in the inequality (15.43) under the

[44] The concept of the mid-year index can be traced to Hill (1998, p. 46):

> When inflation has to be measured over a specified sequence of years, such as a decade, a pragmatic solution to the problems raised above would be to take the middle year as the base year. This can be justified on the grounds that the basket of goods and services purchased in the middle year is likely to be much more representative of the pattern of consumption over the decade as a whole than baskets purchased in either the first or the last years. Moreover, choosing a more representative basket will also tend to reduce, or even eliminate, any bias in the rate of inflation over the decade as a whole as compared with the increase in the CoL index.

Thus, in addition to introducing the concept of a mid-year index, Hill also introduced the terminology *representativity bias*. Baldwin (1990, pp. 255-256) also introduced the term *representativeness*: "Here representativeness [in an index number formula] requires that the weights used in any comparison of price levels are related to the volume of purchases in the periods of comparison."

However, this basic idea dates back to Walsh (1901, p.104;1921a, p. 90). Baldwin (1990, p. 255) also noted that his concept of representativeness was the same as Drechsler's (1973, p. 19) concept of *characteristicity*. For additional material on mid-year indices, see Schultz (1999) and Okamoto (2001). Note that the mid-year index concept could be viewed as a close competitor to Walsh's (1901, p. 431) multi-year fixed basket index where the quantity vector was chosen to be an arithmetic or geometric average of the quantity vectors in the span of periods under consideration.

assumption of long-term trends in prices and normal substitution responses by quantities.

15.50 It is now assumed that the base year quantity vector q^b corresponds to a year that lies between months 0 and t. Under the assumption of long-term trends in prices and normal substitution effects so that there are also long-term trends in quantities (in the opposite direction to the trends in prices so that if the ith commodity price is trending up, then the corresponding ith quantity is trending down), it is likely that the intermediate year quantity vector will lie between the monthly quantity vectors q^0 and q^t. The mid-year Lowe index, $P_{Lo}(p^0,p^t,q^b)$, and the Laspeyres index going from month 0 to t, $P_L(p^0,p^t,q^0)$, will still satisfy the exact relationship given by equation (15.36). Thus $P_{Lo}(p^0,p^t,q^b)$ will equal $P_L(p^0,p^t,q^0)$ plus the covariance term $[\sum_{i=1}^{n} (r_i - r^*)(t_i - t^*)s_i^0\,]/Q_L(q^0,q^b,p^0)$, where $Q_L(q^0,q^b,p^0)$ is the Laspeyres quantity index going from month 0 to t. This covariance term is likely to be negative so that

$$P_L(p^0,p^t,q^0) > P_{Lo}(p^0,p^t,q^b). \qquad (15.44)$$

To see why this covariance is likely to be negative, suppose that there is a long-term upward trend in the price of commodity i so that $r_i - r^* \equiv (p_i^t/p_i^0) - r^*$ is positive. With normal consumer substitution responses, q_i will tend to decrease relatively over time and since q_i^b is assumed to be between q_i^0 and q_i^t, q_i^b/q_i^0 less an average quantity change of this type is likely to be negative. Hence $t_i - t^* \equiv (q_i^b/q_i^0) - t^*$ is likely to be negative. Thus, the covariance is likely to be negative under these circumstances. Therefore, under the assumptions that the quantity base year falls between months 0 and t and that there are long-term trends in prices and normal consumer substitution responses, the Laspeyres index will normally be larger than the corresponding Lowe mid-year index, with the divergence probably growing as month t becomes more distant from month 0.

15.51 It can also be seen that under the above assumptions, the mid-year Lowe index is likely to be greater than the Paasche index between months 0 and t; i.e.,

$$P_{Lo}(p^0,p^t,q^b) > P_P(p^0,p^t,q^t) \qquad (15.45)$$

To see why the above inequality is likely to hold, think of q^b starting at the month 0 quantity vector q^0 and then trending smoothly to the month t quantity vector q^t. When $q^b = q^0$, the Lowe index $P_{Lo}(p^0,p^t,q^b)$ becomes the Laspeyres index $P_L(p^0,p^t,q^0)$. When $q^b = q^t$, the Lowe index $P_{Lo}(p^0,p^t,q^b)$ becomes the Paasche index $P_P(p^0,p^t,q^t)$. Under the assumption of trending prices and normal substitution responses to these trending prices, it was shown earlier that the

Paasche index will be less than the corresponding Laspeyres price index; i.e., that $P_P(p^0,p^t,q^t)$ was less than $P_L(p^0,p^t,q^0)$, recalling the inequality (15.42). Thus, under the assumption of smoothly trending prices and quantities between months 0 and t, and assuming that q^b is between q^0 and q^t, we will have

$$P_P(p^0, p^t, q^t) < P_{Lo}(p^0, p^t, q^b) < P_L(p^0, p^t, q^0) \qquad (15.46)$$

Thus if the base year for the Lowe index is chosen to be in between the base month for the prices, month 0, and the current month for prices, month t, and there are trends in prices with corresponding trends in quantities that correspond to normal consumer substitution effects, then the resulting Lowe index is likely to lie between the Paasche and Laspeyres indices going from months 0 to t. If the trends in prices and quantities are smooth, then choosing the base year half-way between periods 0 and t should give a Lowe index that is approximately half-way between the Paasche and Laspeyres indices; hence it will be very close to an ideal target index between months 0 and t. This basic idea has been implemented by Okamoto (2001), using Japanese consumer data, and he found that the resulting mid-year indices approximated very closely to the corresponding Fisher ideal indices.

15.52 It should be noted that these mid-year indices can only be computed on a retrospective basis; i.e., they cannot be calculated in a timely fashion, as can Lowe indices that use a base year that is prior to month 0. Thus mid-year indices cannot be used to replace the more timely Lowe indices. The above material indicates, however, that these timely Lowe indices are likely to have an upward bias that is even bigger than the usual Laspeyres upward bias compared to an ideal target index, which was taken to be an average of the Paasche and Laspeyres indices.

15.53 All the inequalities derived in this section rest on the assumption of long-term trends in prices (and corresponding economic responses in quantities). If there are no systematic long-run trends in prices, but only random fluctuations around a common trend in all prices, then the above inequalities are not valid and the Lowe index using a prior base year will probably provide a perfectly adequate approximation to both the Paasche and Laspeyres indices. There are, however, reasons for believing that there are some long-run trends in prices. In particular:

- The computer chip revolution of the past 40 years has led to strong downward trends in the prices of products that use these chips intensively. As new uses for chips have been developed over the years, the share of products that are chip intensive has grown and this implies that what used to be a relatively minor problem has become a more major problem.

- Other major scientific advances have had similar effects. For example, the invention of fibre optic cable (and lasers) has led to a downward trend in telecommunications prices as obsolete technologies based on copper wire are gradually replaced.
- Since the end of the Second World War, a series of international trade agreements has dramatically reduced tariffs around the world. These reductions, combined with improvements in transport technologies, have led to a very rapid growth of international trade and remarkable improvements in international specialization. Manufacturing activities in the more developed economies have gradually been outsourced to lower-wage countries, leading to deflation in goods prices in most countries around the world. In contrast, many services cannot be readily outsourced[45] and so, on average, the price of services trends upwards while the price of goods trends downwards.
- At the microeconomic level, there are tremendous differences in growth rates of firms. Successful firms expand their scale, lower their costs, and cause less successful competitors to wither away with their higher prices and lower volumes. This leads to a systematic negative correlation between changes in item prices and the corresponding changes in item volumes that can be very large indeed.

Thus there is some a priori basis for assuming long-run divergent trends in prices. Hence there is some basis for concern that a Lowe index that uses a base year for quantity weights that is prior to the base month for prices may be upwardly biased, compared to a more ideal target index.

The Young index

15.54 Recall the definitions for the base year quantities, q_i^b, and the base year prices, p_i^b, given by equations (15.23) and (15.24) above. The base year expenditure shares can be defined in the usual way as follows:

[45] Some services, however, can be internationally outsourced; e.g., call centres, computer programming and airline maintenance.

$$s_i^b \equiv \frac{p_i^b q_i^b}{\sum_{k=1}^{n} p_k^b q_k^b}; \qquad i = 1,...,n \tag{15.47}$$

Define the vector of base year expenditure shares in the usual way as $s^b \equiv [s_1^b,\ldots,s_n^b]$. These base year expenditure shares were used to provide an alternative formula for the base year *b* Lowe price index going from month 0 to *t*, defined in equation (15.26) as $P_{Lo}(p^0, p^t, q^b) = \left[\sum_{i=1}^{n} s_i^b \left(p_i^t / p_i^b\right)\right] / \left[\sum_{i=1}^{n} s_i^b \left(p_i^0 / p_i^b\right)\right]$. Rather than using this index as their short-run target index, many statistical agencies use the following closely related index:

$$P_Y(p^0, p^t, s^b) \equiv \sum_{i=1}^{n} s_i^b \left(p_i^t / p_i^0\right) \tag{15.48}$$

This type of index was first defined by the English economist, Arthur Young (1812).[46] Note that there is a change in focus when the Young index is used compared to the other indices proposed earlier in this chapter. Up to this point, the indices proposed have been of the fixed basket type (or averages of such indices) where a *commodity basket* that is somehow representative for the two periods being compared is chosen and then "purchased" at the prices of the two periods and the index is taken to be the ratio of these two costs. In contrast, for the Young index, *representative expenditure shares* are chosen that pertain to the two periods under consideration, and then these shares are used to calculate the overall index as a share-weighted average of the individual price ratios, p_i^t/p_i^0. Note that this view of index number theory, based on the share-weighted average of price ratios, is a little different from the view taken at the beginning of this chapter, which saw the index number problem as that of decomposing a value ratio into the product of two terms, one of which expresses the amount of price change between the two periods and the other which expresses the amount of quantity change.[47]

[46] This formula is attributed to Young by Walsh (1901, p. 536; 1932, p. 657).

[47] Fisher's 1922 book is famous for developing the value ratio decomposition approach to index number theory, but his introductory chapters took the share weighted average point of view: "An index number of prices, then shows the *average percentage change* of prices from one point of time to another" (Fisher (1922, p. 3)). Fisher went on to note the importance of economic weighting: "The preceding calculation treats all the commodities as equally important; consequently, the average was called 'simple'. If one commodity is more important than another, we may treat the more important as though it were two or three commodities, thus giving it two or three times as much 'weight' as the other commodity" (Fisher (1922, p. 6)). Walsh (1901, pp. 430-431) considered both approaches: "We can either (1) draw some average of the total money values of the classes during an epoch

15.55 Statistical agencies sometimes regard the Young index, defined above, as an approximation to the Laspeyres price index $P_L(p^0,p^t,q^0)$. Hence, it is of interest to see how the two indices compare. Defining the long-term monthly price relatives going from month 0 to t as $r_i \equiv p_i^t/p_i^0$ and using definitions (15.32) and (15.48):

$$\begin{aligned} P_Y(p^0,p^t,s^b) - P_L(p^0,p^t,q^0) &\equiv \sum_{i=1}^n s_i^b \left(\frac{p_i^t}{p_i^0}\right) - \sum_{i=1}^n s_i^0 \left(\frac{p_i^t}{p_i^0}\right) \\ &= \sum_{i=1}^n \left[s_i^b - s_i^0\right]\left(\frac{p_i^t}{p_i^0}\right) \\ &= \sum_{i=1}^n \left[s_i^b - s_i^0\right] r_i \\ &= \sum_{i=1}^n \left[s_i^b - s_i^0\right]\left[r_i - r^*\right] + r^* \sum_{i=1}^n \left[s_i^b - s_i^0\right] \\ &= \sum_{i=1}^n \left[s_i^b - s_i^0\right]\left[r_i - r^*\right] \end{aligned} \qquad (15.49)$$

since $\sum_{i=1}^n s_i^b = \sum_{i=1}^n s_i^0 = 1$ and using (15.32) which defined $r^* \equiv \sum_{i=1}^n s_i^0 r_i =$ $P_L(p^0,p^t,q^0)$. Thus the Young index $P_Y(p^0,p^t,s^b)$ is equal to the Laspeyres index $P_L(p^0,p^t,q^0)$, plus the *covariance* between the difference in the annual shares pertaining to year b and the month 0 shares, $s_i^b - s_i^0$, and the deviations of the relative prices from their mean, $r_i - r^*$.

15.56 It is no longer possible to guess at what the likely sign of the covariance term is. The question is no longer whether the *quantity* demanded goes down as the price of commodity i goes up (the answer to this question is usually "yes") but the new question is: does the *share* of expenditure go down as the price of commodity i goes up? The answer to this question

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of years, and with weighting so determined employ the geometric average of the price variations [ratios]; or (2) draw some average of the mass quantities of the classes during the epoch, and apply to them Scrope's method." Scrope's method is the same as using the Lowe index. Walsh (1901, pp. 88-90) consistently stressed the importance of weighting price ratios by their economic importance (rather than using equally weighted averages of price relatives). Both the value ratio decomposition approach and the share-weighted average approach to index number theory are studied from the axiomatic perspective in Chapter 16.

depends on the elasticity of demand for the product. Let us provisionally assume, however, that there are long-run trends in commodity prices and if the trend in prices for commodity i is above the mean, then the expenditure share for the commodity trends *down* (and vice versa). Thus we are assuming high elasticities or very strong substitution effects. Assuming also that the base year b is prior to month 0, then under these conditions, suppose that there is a long-term upward trend in the price of commodity i so that $r_i - r^* \equiv (p_i^t/p_i^0) - r^*$ is positive. With the assumed very elastic consumer substitution responses, s_i will tend to decrease relatively over time and since s_i^b is assumed to be prior to s_i^0, s_i^0 is expected to be less than s_i^b or $s_i^b - s_i^0$ will probably be positive. Thus, the covariance is likely to be positive under these circumstances. *Hence with long-run trends in prices and very elastic responses of consumers to price changes, the Young index is likely to be greater than the corresponding Laspeyres index.*

15.57 Assume that there are long-run trends in commodity prices. If the trend in prices for commodity i is above the mean, then suppose that the expenditure share for the commodity trends *up* (and vice versa). Thus we are assuming low elasticities or very weak substitution effects. Assume also that the base year b is prior to month 0 and suppose that there is a long-term upward trend in the price of commodity i so that $r_i - r^* \equiv (p_i^t/p_i^0) - r^*$ is positive. With the assumed very inelastic consumer substitution responses, s_i will tend to increase relatively over time and since s_i^b is assumed to be prior to s_i^0, it will be the case that s_i^0 is greater than s_i^b or $s_i^b - s_i^0$ is negative. Thus, the covariance is likely to be negative under these circumstances. *Hence with long-run trends in prices and very inelastic responses of consumers to price changes, the Young index is likely to be less than the corresponding Laspeyres index.*

15.58 The previous two paragraphs indicate that, a priori, it is not known what the likely difference between the Young index and the corresponding Laspeyres index will be. If elasticities of substitution are close to one, then the two sets of expenditure shares, s_i^b and s_i^0, will be close to each other and the difference between the two indices will be close to zero. If monthly expenditure shares have strong seasonal components, however, then the annual shares s_i^b could differ substantially from the monthly shares s_i^0.

15.59 It is useful to have a formula for updating the previous month's Young price index using just month-over-month price relatives. The Young index for month t+1, $P_Y(p^0,p^{t+1},s^b)$, can be written in terms of the Young index for month t, $P_Y(p^0,p^t,s^b)$, and an updating factor as follows:

$$P_Y(p^0, p^{t+1}, s^b) \equiv \sum_{i=1}^{n} s_i^b \left(\frac{p_i^{t+1}}{p_i^0} \right)$$

$$= P_Y(p^0, p^t, s^b) \frac{\sum_{i=1}^{n} s_i^b \left(p_i^{t+1} / p_i^0 \right)}{\sum_{i=1}^{n} s_i^b \left(p_i^t / p_i^0 \right)}$$

$$= P_Y(p^0, p^t, s^b) \frac{\sum_{i=1}^{n} p_i^b q_i^b \left(p_i^{t+1} / p_i^0 \right)}{\sum_{i=1}^{n} p_i^b q_i^b \left(p_i^t / p_i^0 \right)}$$

using definition (15.47)

$$= P_Y(p^0, p^t, s^b) \frac{\sum_{i=1}^{n} p_i^b q_i^b \left(\frac{p_i^t}{p_i^0} \right) \left(\frac{p_i^{t+1}}{p_i^t} \right)}{\sum_{i=1}^{n} p_i^b q_i^b (p_i^t / p_i^0)}$$

$$= P_Y(p^0, p^t, s^b) \left[\sum_{i=1}^{n} s_i^{b0t} (p_i^{t+1} / p_i^t) \right] \qquad (15.50)$$

where the hybrid weights s_i^{b0t} are defined by

$$s_i^{b0t} \equiv \frac{p_i^b q_i^b \left(p_i^t / p_i^0 \right)}{\sum_{k=1}^{n} p_k^b q_k^b \left(p_k^t / p_k^0 \right)} = \frac{s_i^b \left(p_i^t / p_i^0 \right)}{\sum_{k=1}^{n} s_k^b \left(p_k^t / p_k^0 \right)} \qquad i = 1,\ldots,n \qquad (15.51)$$

Thus the hybrid weights s_i^{b0t} can be obtained from the base year weights s_i^b by updating them; i.e., by multiplying them by the price relatives (or *indices* at higher levels of aggregation), p_i^t/p_i^0. Thus the required updating factor, going from month t to month t+1, is the chain link index, $\sum_{i=1}^{n} s_i^{b0t}(p_i^{t+1}/p_i^t)$, which uses the hybrid share weights s_i^{b0t} defined by equation (15.51).

15.60 Even if the Young index provides a close approximation to the corresponding Laspeyres index, it is difficult to recommend the use of the Young index as a final estimate of the change in prices going from period 0 to *t*, just as it was difficult to recommend the use of the Laspeyres index as the *final* estimate of inflation going from period 0 to *t*. Recall that the problem with the Laspeyres index was its lack of symmetry in the treatment of the two periods under consideration; i.e., using the justification for the Laspeyres index as a good fixed basket index, there was an identical justification for the use of the Paasche index as an equally good fixed basket index to compare periods 0 and *t*. The Young index suffers from a similar lack of symmetry with respect to the treatment of the base period. The problem can be explained as follows. The Young index, $P_Y(p^0,p^t,s^b)$ defined by equation (15.48) calculates the price change between months 0 and *t* treating month 0 as the base. But there is no particular reason to necessarily treat month 0 as the base month other than convention. Hence, if we treat month *t* as the base and use the same formula to measure the price change from month *t* back to month 0, the index $P_Y(p^t,p^0,s^b) = \sum_{i=1}^{n} s_i^b\left(p_i^0/p_i^t\right)$ would be appropriate. This estimate of price change can then be made comparable to the original Young index by taking its reciprocal, leading to the following *rebased Young index*[48], $P_Y^*(p^0,p^t,s^b)$, defined as

$$P_Y^*(p^0,p^t,s^b) \equiv 1\Big/\sum_{i=1}^{n} s_i^b(p_i^0/p_i^t)$$
$$= \left[\sum_{i=1}^{n} s_i^b(p_i^t/p_i^0)^{-1}\right]^{-1} \qquad (15.52)$$

The rebased Young index, $P_Y^*(p^0,p^t,s^b)$, which uses the current month as the initial base period, is a *share-weighted harmonic mean* of the price relatives going from month 0 to

[48] Using Fisher's (1922, p. 118) terminology, $P_Y^*(p^0,p^t,s^b) \equiv 1/[P_Y(p^t,p^0,s^b)]$ is the *time antithesis* of the original Young index, $P_Y(p^0,p^t,s^b)$.

month t, whereas the original Young index, $P_Y(p^0,p^t,s^b)$, is a *share-weighted arithmetic mean* of the same price relatives.

15.61 Fisher argued as follows that an index number formula should give the same answer no matter which period was chosen as the base:

> Either one of the two times may be taken as the "base". Will it make a difference which is chosen? Certainly, it *ought* not and our Test 1 demands that it shall not. More fully expressed, the test is that the formula for calculating an index number should be such that it will give the same ratio between one point of comparison and the other point, *no matter which of the two is taken as the base* (Fisher (1922, p. 64)).

15.62 The problem with the Young index is that not only does it not coincide with its rebased counterpart, but there is a definite inequality between the two indices, namely:

$$P_Y^*(p^0, p^t, s^b) \le P_Y(p^0, p^t, s^b) \qquad (15.53)$$

with a strict inequality provided that the period t price vector p^t is not proportional to the period 0 price vector p^0.[49] A statistical agency that uses the direct Young index $P_Y(p^0,p^t,s^b)$ will generally show a higher inflation rate than a statistical agency that uses the same raw data but uses the rebased Young index, $P_Y^*(p^0,p^t,s^b)$.

15.63 The inequality (15.53) does not tell us by how much the Young index will exceed its rebased time antithesis. In Appendix 15.3, however, it is shown that to the accuracy of a certain second-order Taylor series approximation, the following relationship holds between the direct Young index and its time antithesis:

$$P_Y(p^0, p^t, s^b) \approx P_Y^*(p^0, p^t, s^b) + P_Y(p^0, p^t, s^b)\ \mathrm{Var}\ e \qquad (15.54)$$

where Var e is defined as

[49] These inequalities follow from the fact that a harmonic mean of M positive numbers is always equal to or less than the corresponding arithmetic mean; see Walsh (1901, p.517) or Fisher (1922, pp. 383-384). This inequality is a special case of Schlömilch's (1858) inequality; see Hardy, Littlewood and Polya (1934, p. 26). Walsh (1901, pp. 330-332) explicitly noted the inequality (15.53) and also noted that the corresponding geometric average would fall between the harmonic and arithmetic averages. Walsh (1901, p. 432) computed some numerical examples of the Young index and found big differences between it and his "best" indices, even using weights that were representative for the periods being compared. Recall that the Lowe index becomes the Walsh index when geometric mean quantity weights are chosen and so the Lowe index can perform well when representative weights are used. This is not necessarily the case for the Young index, even using representative weights. Walsh (1901, p. 433) summed up his numerical experiments with the Young index as follows: "In fact, Young's method, in every form, has been found to be bad."

$$\text{Var } e \equiv \sum_{i=1}^{n} s_i^b \left[e_i - e^* \right]^2 \tag{15.55}$$

The deviations e_i are defined by $1+e_i = r_i/r^*$ for $i = 1,\ldots, n$ where the r_i and their weighted mean r^* are defined by

$$r_i \equiv p_i^t / p_i^0; \qquad i = 1,\ldots,n; \tag{15.56}$$

$$r^* \equiv \sum_{i=1}^{n} s_i^b r_i \tag{15.57}$$

which turns out to equal the direct Young index, $P_Y(p^0,p^t,s^b)$. The weighted mean of the e_i is defined as

$$e^* \equiv \sum_{i=1}^{n} s_i^b e_i \tag{15.58}$$

which turns out to equal 0. *Hence the more dispersion there is in the price relatives p_i^t/p_i^0, to the accuracy of a second-order approximation, the more the direct Young index will exceed its counterpart that uses month t as the initial base period rather than month 0.*

15.64 Given two a priori equally plausible index number formulae that give different answers, such as the Young index and its time antithesis, Fisher (1922, p. 136) generally suggested taking the geometric average of the two indices.[50] A benefit of this averaging is that the resulting formula will satisfy the time reversal test. Thus rather than using *either* the base period 0 Young index, $P_Y(p^0,p^t,s^b)$, *or* the current period t Young index, $P_Y{}^*(p^0,p^t,s^b)$, which is always below the base period 0 Young index if there is any dispersion in relative prices, it seems preferable to use the following index, which is the *geometric average* of the

[50] "We now come to a third use of these tests, namely, to 'rectify' formulae, i.e., to derive from any given formula which does not satisfy a test another formula which does satisfy it; …. This is easily done by 'crossing', that is, by averaging antitheses. If a given formula fails to satisfy Test 1 [the time reversal test], its time antithesis will also fail to satisfy it; but the two will fail, as it were, in opposite ways, so that a cross between them (obtained by *geometrical* averaging) will give the golden mean which does satisfy" (Fisher (1922, p. 136)).

Actually the basic idea behind Fisher's rectification procedure was suggested by Walsh, who was a discussant for Fisher (1921), where Fisher gave a preview of his 1922 book: "We merely have to take any index number, find its antithesis in the way prescribed by Professor Fisher, and then draw the geometric mean between the two" (Walsh (1921b, p. 542)).

two alternatively based Young indices.[51]

$$P_Y^{**}(p^0, p^t, s^b) \equiv \left[P_Y(p^0, p^t, s^b) P_Y^*(p^0, p^t, s^b)\right]^{1/2} \qquad (15.59)$$

If the base year shares s_i^b happen to coincide with both the month 0 and month t shares, s_i^0 and s_i^t respectively, it can be seen that the time-rectified Young index $P_Y^{**}(p^0,p^t,s^b)$ defined by equation (15.59) will coincide with the Fisher ideal price index between months 0 and t, $P_F(p^0,p^t,q^0,q^t)$ (which will also equal the Laspeyres and Paasche indices under these conditions). Note also that the index P_Y^{**} defined by equation (15.59) can be produced on a timely basis by a statistical agency.

The Divisia index and discrete approximations to it

The Divisia price and quantity indices

15.65 The second broad approach to index number theory relies on the assumption that price and quantity data change in a more or less continuous way.

15.66 Suppose that the price and quantity data on the n commodities in the chosen domain of definition can be regarded as continuous functions of (continuous) time, say $p_i(t)$ and $q_i(t)$ for i = 1,…,n. The value of consumer expenditure at time t is $V(t)$ defined in the obvious way as:

$$V(t) \equiv \sum_{i=1}^{n} p_i(t) q_i(t) \qquad (15.60)$$

15.67 Now suppose that the functions $p_i(t)$ and $q_i(t)$ are differentiable. Then both sides of the definition (15.60) can be differentiated with respect to time to obtain:

$$V'(t) = \sum_{i=1}^{n} p_i'(t) q_i(t) + \sum_{i=1}^{n} p_i(t) q_i'(t) \qquad (15.61)$$

Divide both sides of equation (15.61) through by $V(t)$ and using definition (15.60), the following equation is obtained:

[51] This index is a base year weighted counterpart to an equally weighted index proposed by Carruthers, Sellwood and Ward (1980, p. 25) and Dalén (1992, p. 140) in the context of elementary index number formulae. See Chapter 20 for further discussion of this unweighted index.

$$\frac{V'(t)}{V(t)} = \frac{\sum_{i=1}^{n} p_i'(t)q_i(t) + \sum_{i=1}^{n} p_i(t)q_i'(t)}{\sum_{j=1}^{n} p_j(t)q_j(t)}$$

$$= \sum_{i=1}^{n} \frac{p_i'(t)}{p_i(t)} s_i(t) + \sum_{i=1}^{n} \frac{q_i'(t)}{q_i(t)} s_i(t) \qquad (15.62)$$

where the time t expenditure share on commodity i, $s_i(t)$, is defined as:

$$s_i(t) \equiv \frac{p_i(t)q_i(t)}{\sum_{m=1}^{n} p_m(t)q_m(t)} \quad \text{for } i = 1,2,\ldots,n \qquad (15.63)$$

15.68 Divisia (1926, p. 39) argued as follows: *suppose* the aggregate value at time t, $V(t)$, can be written as the product of a time t price level function, $P(t)$ say, times a time t quantity level function, $Q(t)$ say; i.e., we have:

$$V(t) = P(t)Q(t) \qquad (15.64)$$

Suppose further that the functions $P(t)$ and $Q(t)$ are differentiable. Then differentiating the equation (15.64) yields:

$$V'(t) = P'(t)Q(t) + P(t)Q'(t) \qquad (15.65)$$

Dividing both sides of equation (15.65) by $V(t)$ and using equation (15.64) leads to the following equation:

$$\frac{V'(t)}{V(t)} = \frac{P'(t)}{P(t)} + \frac{Q'(t)}{Q(t)} \qquad (15.66)$$

15.69 Divisia compared the two expressions for the logarithmic value derivative, $V'(t)/V(t)$, given by equations (15.62) and (15.66), and he simply defined the logarithmic rate of change of the *aggregate price level*, $P'(t)/P(t)$, as the first set of terms on the right-hand side of (15.62). He also simply defined the logarithmic rate of change of the *aggregate quantity level*, $Q'(t)/Q(t)$, as the second set of terms on the right-hand side of equation (15.62). That is, he made the following definitions:

$$\frac{P'(t)}{P(t)} \equiv \sum_{i=1}^{n} s_i(t) \frac{p_i^{'}(t)}{p_i(t)} \qquad (15.67)$$

$$\frac{Q'(t)}{Q(t)} \equiv \sum_{i=1}^{n} s_i(t) \frac{q_i^{'}(t)}{q_i(t)} \qquad (15.68)$$

15.70 Definitions (15.67) and (15.68) are reasonable definitions for the proportional changes in the aggregate price and quantity (or quantity) levels, $P(t)$ and $Q(t)$.[52] The problem with these definitions is that economic data are not collected in *continuous* time; they are collected in *discrete* time. In other words, even though transactions can be thought of as occurring in continuous time, no consumer records his or her purchases as they occur in continuous time; rather, purchases over a finite time period are cumulated and then recorded. A similar situation occurs for producers or sellers of commodities; firms cumulate their sales over discrete periods of time for accounting or analytical purposes. If it is attempted to approximate continuous time by shorter and shorter discrete time intervals, empirical price and quantity data can be expected to become increasingly erratic since consumers only make purchases at discrete points of time (and producers or sellers of commodities only make sales at discrete points of time). It is, however, still of some interest to approximate the continuous time price and quantity levels, $P(t)$ and $Q(t)$ defined implicitly by equations (15.67) and (15.68), by discrete time approximations. This can be done in two ways. Either methods of numerical approximation can be used or assumptions can be made about the path taken through time by the functions $p_i(t)$ and $q_i(t)$ ($i = 1,\ldots, n$). The first strategy is used in the following section. For discussions of the second strategy, see Vogt (1977; 1978), Van Ijzeren (1987, pp. 8-12), Vogt and Barta (1997) and Balk (2000a).

15.71 There is a connection between the Divisia price and quantity levels, $P(t)$ and $Q(t)$, and the economic approach to index number theory. This connection is, however, best made after studying the economic approach to index number theory. Since this material is rather technical, it has been relegated to Appendix 15.4.

[52] If these definitions are applied (approximately) to the Young index studied in the previous section, then it can be seen that in order for the Young price index to be consistent with the Divisia price index, the base year shares should be chosen to be average shares that apply to the entire time period between months 0 and *t*.

Discrete approximations to the continuous time Divisia index

15.72 In order to make operational the continuous time Divisia price and quantity levels, $P(t)$ and $Q(t)$ defined by the differential equations (15.67) and (15.68), it is necessary to convert to discrete time. Divisia (1926, p. 40) suggested a straightforward method for doing this conversion, which we now outline.

15.73 Define the following price and quantity (forward) differences:

$$\Delta P \equiv P(1) - P(0) \qquad (15.69)$$

$$\Delta p_i \equiv p_i(1) - p_i(0); \quad i = 1,...,n \qquad (15.70)$$

Using the above definitions:

$$\frac{P(1)}{P(0)} = \frac{P(0)+\Delta P}{P(0)} = 1 + \frac{\Delta P}{P(0)} \approx 1 + \frac{\sum_{i=1}^{n} \Delta p_i q_i(0)}{\sum_{m=1}^{n} p_m(0) q_m(0)}$$

using (15.67) when $t = 0$ and approximating $p_i^{'}(0)$ by the difference Δp_i

$$= \frac{\sum_{i=1}^{n} \{p_i(0) + \Delta p_i\} q_i(0)}{\sum_{m=1}^{n} p_m(0) q_m(0)} = \frac{\sum_{i=1}^{n} p_i(1) q_i(0)}{\sum_{m=1}^{n} p_m(0) q_m(0)} = P_L(p^0, p^1, q^0, q^1) \qquad (15.71)$$

where $p^t \equiv [p_1(t),\ldots,p_n(t)]$ and $q^t \equiv [q_1(t),\ldots,q_n(t)]$ for $t = 0,1$. Thus, it can be seen that Divisia's discrete approximation to his continuous time price index is just the Laspeyres price index, P_L, defined above by equation (15.5).

15.74 But now a problem noted by Frisch (1936, p. 8) occurs: instead of approximating the derivatives by the discrete (forward) differences defined by equations (15.69) and (15.70), other approximations could be used and a wide variety of discrete time approximations could be obtained. For example, instead of using forward differences and evaluating the index at time $t = 0$, it would be possible to use backward differences and evaluate the index at time $t = 1$. These backward differences are defined as:

$$\Delta_b p_i \equiv p_i(0) - p_i(1); \quad i = 1, \ldots, n \tag{15.72}$$

This use of backward differences leads to the following approximation for $P(0)/P(1)$:

$$\frac{P(0)}{P(1)} = \frac{P(1) + \Delta_b P}{P(1)} = 1 + \frac{\Delta_b P}{P(1)} \approx 1 + \frac{\sum_{i=1}^{n} \Delta_b p_i q_i(1)}{\sum_{m=1}^{n} p_m(1) q_m(1)}$$

using (15.67) when $t = 1$ and approximating $p_i'(1)$ by the difference $\Delta_b p_i$

$$= \frac{\sum_{i=1}^{n} \{p_i(1) + \Delta_b p_i\} q_i(1)}{\sum_{m=1}^{n} p_m(1) q_m(1)} = \frac{\sum_{i=1}^{n} p_i(0) q_i(1)}{\sum_{m=1}^{n} p_m(1) q_m(1)} = \frac{1}{P_P(p^0, p^1, q^0, q^1)} \tag{15.73}$$

where P_P is the Paasche index defined above by equation (15.6). Taking reciprocals of both sides of equation (15.73) leads to the following discrete approximation to $P(1)/P(0)$:

$$\frac{P(1)}{P(0)} \approx P_P \tag{15.74}$$

15.75 Thus, as Frisch[53] noted, both the Paasche and Laspeyres indices can be regarded as (equally valid) approximations to the continuous time Divisia price index.[54] Since the Paasche and Laspeyres indices can differ considerably in some empirical applications, it can be seen that Divisia's idea is not all that helpful in determining a *unique* discrete time index number formula.[55] What is useful about the Divisia indices is the idea that as the discrete

[53] "As the elementary formula of the chaining, we may get Laspeyres' or Paasche's or Edgeworth's or nearly any other formula, according as we choose the approximation principle for the steps of the numerical integration" (Frisch (1936, p. 8)).

[54] Diewert (1980, p. 444) also obtained the Paasche and Laspeyres approximations to the Divisia index, using a somewhat different approximation argument. He also showed how several other popular discrete time index number formulae could be regarded as approximations to the continuous time Divisia index.

[55] Trivedi (1981) systematically examined the problems involved in finding a "best" discrete time approximation to the Divisia indices using the techniques of numerical analysis. These numerical analysis techniques depend on the assumption that the "true" continuous time micro-price functions, $p_i(t)$, can be adequately represented by a polynomial approximation. Thus we are led to the conclusion that the "best" discrete time approximation to the Divisia index depends on assumptions that are difficult to verify.

unit of time gets smaller, discrete approximations to the Divisia indices can approach meaningful economic indices under certain conditions. Moreover, if the Divisia concept is accepted as the "correct" one for index number theory, then the corresponding "correct" discrete time counterpart might be taken as a weighted average of the chain price relatives pertaining to the adjacent periods under consideration, where the weights are somehow representative of the two periods under consideration.

Fixed base versus chain indices

15.76 In this section[56], we discuss the merits of using the chain system for constructing price indices in the time series context versus using the fixed base system.[57]

15.77 The chain system[58] measures the change in prices going from one period to a subsequent period using a bilateral index number formula involving the prices and quantities pertaining to the two adjacent periods. These one-period rates of change (the links in the chain) are then cumulated to yield the relative levels of prices over the entire period under consideration. Thus if the bilateral price index is *P*, the chain system generates the following pattern of price levels for the first three periods:

$$1, \quad P(p^0, p^1, q^0, q^1), \quad P(p^0, p^1, q^0, q^1)P(p^1, p^2, q^1, q^2) \qquad (15.75)$$

15.78 In contrast, the fixed base system of price levels, using the same bilateral index number formula *P*, simply computes the level of prices in period *t* relative to the base period 0 as $P(p^0,p^t,q^0,q^t)$. Thus the fixed base pattern of price levels for periods 0,1 and 2 is:

$$1, P(p^0, p^1, q^0, q^1), \quad P(p^0, p^2, q^0, q^2) \qquad (15.76)$$

15.79 Note that in both the chain system and the fixed base system of price levels defined

[56] This section is largely based on the work of Hill (1988; 1993, p.385-390).

[57] The results in Appendix 15.4 provide some theoretical support for the use of chain indices in that it is shown that under certain conditions, the Divisia index will equal an economic index. Hence any discrete approximation to the Divisia index will approach the economic index as the time period gets shorter. Thus under certain conditions, chain indices will approach an underlying economic index.

[58] The chain principle was introduced independently into the economics literature by Lehr (1885. pp. 45-46) and Marshall (1887, p. 373). Both authors observed that the chain system would mitigate the difficulties arising from the introduction of new commodities into the economy, a point also mentioned by Hill (1993, p. 388). Fisher (1911, p. 203) introduced the term "chain system".

by the formulae (15.75) and (15.76), the base period price level is set equal to 1. The usual practice in statistical agencies is to set the base period price level equal to 100. If this is done, then it is necessary to multiply each of the numbers in the formulae (15.75) and (15.76) by 100.

15.80 Because of the difficulties involved in obtaining current period information on quantities (or equivalently, on expenditures), many statistical agencies loosely base their consumer price index on the use of the Laspeyres formula (15.5) and the fixed base system. Therefore, it is of interest to look at some of the possible problems associated with the use of fixed base Laspeyres indices.

15.81 The main problem with the use of fixed base Laspeyres indices is that the period 0 fixed basket of commodities that is being priced out in period t can often be quite different from the period t basket. Thus if there are systematic trends in at least some of the prices and quantities[59] in the index basket, the fixed base Laspeyres price index $P_L(p^0,p^t,q^0,q^t)$ can be quite different from the corresponding fixed base Paasche price index, $P_P(p^0,p^t,q^0,q^t)$.[60] This means that both indices are likely to be an inadequate representation of the movement in average prices over the time period under consideration.

15.82 The fixed base Laspeyres quantity index cannot be used for ever: eventually, the current period quantities q^t are so far removed from the base period quantities q^0 that the base must be changed. Chaining is merely the limiting case where the base is changed each period.

15.83 The main advantage of the chain system is that under normal conditions, chaining will reduce the spread between the Paasche and Laspeyres indices.[61] These two indices each provide an asymmetric perspective on the amount of price change that has occurred between the two periods under consideration and it could be expected that a single point estimate of

[59] Examples of rapidly downward trending prices and upward trending quantities are computers, electronic equipment of all types, Internet access and telecommunication charges.

[60] Note that $P_L(p^0,p^t,q^0,q^t)$ will equal $P_P(p^0,p^t,q^0,q^t)$ if *either* the two quantity vectors q^0 and q^t are proportional *or* the two price vectors p^0 and p^t are proportional. Thus in order to obtain a difference between the Paasche and Laspeyres indices, nonproportionality in *both* prices and quantities is required.

[61] See Diewert (1978, p. 895) and Hill (1988; 1993, pp. 387-388).

the aggregate price change should lie between these two estimates. Thus the use of either a chained Paasche or Laspeyres index will usually lead to a smaller difference between the two and hence to estimates that are closer to the "truth".[62]

15.84 Hill (1993, p. 388), drawing on the earlier research of Szulc (1983) and Hill (1988, pp. 136-137), noted that it is not appropriate to use the chain system when prices oscillate or bounce. This phenomenon can occur in the context of regular seasonal fluctuations or in the context of price wars. However, in the context of roughly monotonically changing prices and quantities, Hill (1993, p. 389) recommended the use of chained symmetrically weighted indices (see paragraphs 15.18 to 15.32). The Fisher and Walsh indices are examples of symmetrically weighted indices.[63]

15.85 It is possible to be a little more precise about the conditions under which to chain or not to chain. Basically, chaining is advisable if the prices and quantities pertaining to adjacent periods are *more similar* than the prices and quantities of more distant periods, since this strategy will lead to a narrowing of the spread between the Paasche and Laspeyres indices at each link.[64] Of course, one needs a measure of how similar are the prices and quantities

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[62] This observation will be illustrated with an artificial data set in Chapter 19.

[63] Regular seasonal fluctuations can cause monthly or quarterly data to "bounce" – using the term coined by Szulc (1983, p. 548) – and chaining bouncing data can lead to a considerable amount of index "drift"; i.e., if after 12 months, prices and quantities return to their levels of a year earlier, then a chained monthly index will usually not return to unity. Hence, the use of chained indices for "noisy" monthly or quarterly data is not recommended without careful consideration.

[64] Walsh, in discussing whether fixed base or chained index numbers should be constructed, took for granted that the precision of all reasonable bilateral index number formulae would improve, provided that the two periods or situations being compared were more similar, and hence favoured the use of chained indices: "The question is really, in which of the two courses [fixed base or chained index numbers] are we likely to gain greater exactness in the comparisons actually made? Here the probability seems to incline in favor of the second course; for the conditions are likely to be less diverse between two contiguous periods than between two periods say fifty years apart" (Walsh (1901, p. 206)).

Walsh (1921a, pp. 84-85) later reiterated his preference for chained index numbers. Fisher also made use of the idea that the chain system would usually make bilateral comparisons between price and quantity data that were more similar, and hence the resulting comparisons would be more accurate:

> The index numbers for 1909 and 1910 (each calculated in terms of 1867-1877) are compared with each other. But direct comparison between 1909 and 1910 would give a different and more valuable result. To use a common base is like comparing the relative heights of two men by measuring the height of each above the floor, instead of putting them back to back and directly measuring the difference of level between the tops of their heads (Fisher (1911, p. 204)).
>
> It seems, therefore, advisable to compare each year with the next, or, in other words, to make each year the base year for the next. Such a procedure has been recommended by Marshall, Edgeworth and Flux. It largely meets the difficulty of non-uniform changes in the Q's, for any inequalities for successive years are relatively small (Fisher

pertaining to two periods. The similarity measures could be *relative* ones or *absolute* ones. In the case of absolute comparisons, two vectors of the same dimension are similar if they are identical and dissimilar otherwise. In the case of relative comparisons, two vectors are similar if they are proportional and dissimilar if they are non-proportional.[65] Once a similarity measure has been defined, the prices and quantities of each period can be compared to each other using this measure, and a "tree" or path that links all of the observations can be constructed where the most similar observations are compared with each other using a bilateral index number formula.[66] Hill (1995) defined the price structures between two countries to be more dissimilar the bigger the spread between P_L and P_P; i.e., the bigger is $\{P_L/P_P, P_P/P_L\}$. The problem with this measure of dissimilarity in the price structures of the two countries is that it could be the case that $P_L = P_P$ (so that the Hill measure would register a maximal degree of similarity), but p^0 could be very different from p^t. Thus there is a need for a more systematic study of similarity (or dissimilarity) measures in order to pick the "best" one that could be used as an input into Hill's (1999a; 1999b; 2001) spanning tree algorithm for linking observations.

15.86 The method of linking observations explained in the previous paragraph, based on the similarity of the price and quantity structures of any two observations, may not be practical in a statistical agency context since the addition of a new period may lead to a reordering of the previous links. The above "scientific" method for linking observations may be useful, however, in deciding whether chaining is preferable or whether fixed base indices should be used while making month-to-month comparisons within a year.

15.87 Some index number theorists have objected to the chain principle on the grounds that it has no counterpart in the spatial context:

> They [chain indices] only apply to intertemporal comparisons, and in contrast to direct indices they are not applicable to cases in which no natural order or sequence exists. Thus the idea of a chain index for

(1911, pp. 423-424)).

[65] Diewert (2002b) takes an axiomatic approach to defining various *indices* of absolute and relative dissimilarity.

[66] Fisher (1922, pp.271-276) hinted at the possibility of using spatial linking; i.e., of linking countries that are similar in structure. The modern literature has, however, grown as a result of the pioneering efforts of Robert Hill (1995; 1999a; 1999b; 2001). Hill (1995) used the spread between the Paasche and Laspeyres price indices as an indicator of similarity, and showed that this criterion gives the same results as a criterion that looks at the spread between the Paasche and Laspeyres quantity indices.

example has no counterpart in interregional or international price comparisons, because countries cannot be sequenced in a "logical" or "natural" way (there is no k+1 nor k–1country to be compared with country k) (von der Lippe (2001, p. 12)).[67]

This is of course correct, but the approach of Hill does lead to a "natural" set of spatial links. Applying the same approach to the time series context will lead to a set of links between periods which may not be month-to-month but it will in many cases justify year-over-year linking of the data pertaining to the same month. This problem is reconsidered in Chapter 22.

15.88 It is of some interest to determine if there are index number formulae that give the same answer when either the fixed base or chain system is used. Comparing the sequence of chain indices defined by the expression (15.75) to the corresponding fixed base indices, it can be seen that we will obtain the same answer in all three periods if the index number formula P satisfies the following functional equation for all price and quantity vectors:

$$P(p^0,p^2,q^0,q^2)=P(p^0,p^1,q^0,q^1)P(p^1,p^2,q^1,q^2) \qquad (15.77)$$

If an index number formula P satisfies the equation (15.77), then P satisfies the *circularity test*.[68]

15.89 If it is assumed that the index number formula P satisfies certain properties or tests in addition to the circularity test above,[69] then Funke, Hacker and Voeller (1979) showed that P must have the following functional form, originally established by Konüs and Byushgens[70] (1926, pp. 163-166):[71]

[67] It should be noted that von der Lippe (2001, pp. 56-58) is a vigorous critic of all index number tests based on symmetry in the time series context, although he is willing to accept symmetry in the context of making international comparisons. "But there are good reasons *not* to insist on such criteria in the *intertemporal* case. When no symmetry exists between 0 and t, there is no point in interchanging 0 and t" (von der Lippe (2001, p. 58)).

[68] The test name is attributable to Fisher (1922, p. 413) and the concept originated from Westergaard (1890, pp. 218-219).

[69] The additional tests referred to above are: (i) positivity and continuity of $P(p^0,p^1,q^0,q^1)$ for all strictly positive price and quantity vectors p^0,p^1,q^0,q^1; (ii) the identity test; (iii) the commensurability test; (iv) $P(p^0,p^1,q^0,q^1)$ is positively homogeneous of degree one in the components of p^1, and (v) $P(p^0,p^1,q^0,q^1)$ is positively homogeneous of degree zero in the components of q^1.

[70] Konüs and Byushgens show that the index defined by equation (15.78) is exact for Cobb-Douglas (1928) preferences; see also Pollak (1983, pp. 119-120). The concept of an exact index number formula is explained in Chapter 17.

[71] The result in equation (15.78) can be derived using results in Eichhorn (1978, pp. 167-168) and Vogt and Barta (1997, p. 47). A simple proof can be found in Balk (1995). This result vindicates Irving Fisher's (1922, p. 274) intuition that "the only formulae which conform perfectly to the circular test are index numbers which have

$$P_{KB}(p^0, p^1, q^0, q^1) \equiv \prod_{i=1}^{n} \left(\frac{p_i^1}{p_i^0} \right)^{\alpha_i} \qquad (15.78)$$

where the n constants α_i satisfy the following restrictions:

$$\sum_{i=1}^{n} \alpha_i = 1 \quad \text{and} \quad \alpha_i > 0 \quad \text{for} \quad i = 1,...,n \qquad (15.79)$$

Thus under very weak regularity conditions, the only price index satisfying the circularity test is a weighted geometric average of all the individual price ratios, the weights being constant through time.

15.90 An interesting special case of the family of indices defined by equation (15.78) occurs when the weights α_i are all equal. In this case, P_{KB} reduces to the Jevons (1865) index:

$$P_J(p^0, p^1, q^0, q^1) \equiv \prod_{i=1}^{n} \left(\frac{p_i^1}{p_i^0} \right)^{\frac{1}{n}} \qquad (15.80)$$

15.91 The problem with the indices defined by Konüs and Byushgens, and Jevons is that the individual price ratios, p_i^1/p_i^0, have weights (either α_i or $1/n$) that are *independent* of the economic importance of commodity i in the two periods under consideration. Put another way, these price weights are independent of the quantities of commodity i consumed or the expenditures on commodity i during the two periods. Hence, these indices are not really suitable for use by statistical agencies at higher levels of aggregation when expenditure share information is available.[72]

15.92 The above results indicate that it is not useful to ask that the price index P satisfy the

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constant weights…". Fisher (1922, p. 275) went on to assert: "But, clearly, constant weighting is not theoretically correct. If we compare 1913 with 1914, we need one set of weights; if we compare 1913 with 1915, we need, theoretically at least, another set of weights. … Similarly, turning from time to space, an index number for comparing the United States and England requires one set of weights, and an index number for comparing the United States and France requires, theoretically at least, another."

[72] When there are only two periods being compared and expenditure share information is available for both periods, then the economic approach will suggest in Chapter 17 that good choices for the weights α_i are the arithmetic averages of the period 0 and 1 expenditure shares, s_i^0 and s_i^1.

circularity test *exactly*. It is nevertheless of some interest to find index number formulae that satisfy the circularity test to some degree of approximation, since the use of such an index number formula will lead to measures of aggregate price change that are more or less the same no matter whether we use the chain or fixed base systems. Fisher (1922, p. 284) found that deviations from circularity using his data set and the Fisher ideal price index P_F defined by equation (15.12) above were quite small. This relatively high degree of correspondence between fixed base and chain indices has been found to hold for other symmetrically weighted formulae, such as the Walsh index P_W defined by equation (15.19).[73] In most time series applications of index number theory where the base year in fixed base indices is changed every five years or so, it will not matter very much whether the statistical agency uses a fixed base price index or a chain index, provided that a symmetrically weighted formula is used.[74] The choice between a fixed base price index or chain index will depend, of course, on the length of the time series considered and the degree of variation in the prices and quantities as we go from period to period. The more prices and quantities are subject to large fluctuations (rather than smooth trends), the less the correspondence.[75]

15.93 It is possible to give a theoretical explanation for the approximate satisfaction of the circularity test for symmetrically weighted index number formulae. Another symmetrically weighted formula is the Törnqvist index P_T.[76] The natural logarithm of this index is defined as follows:

$$\ln P_T(p^0, p^1, q^0, q^1) \equiv \sum_{i=1}^{n} \frac{1}{2}\left(s_i^0 + s_i^1\right)\ln\left(\frac{p_i^1}{p_i^0}\right) \qquad (15.81)$$

where the period t expenditure shares s_i^t are defined by equation (15.7). Alterman, Diewert

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[73] See, for example, Diewert (1978, p. 894)). Walsh (1901, pp. 424 and 429) found that his three preferred formulae all approximated each other very well, as did the Fisher ideal for his artificial data set.

[74] More specifically, most superlative indices (which are symmetrically weighted) will satisfy the circularity test to a high degree of approximation in the time series context. See Chapter 17 for the definition of a superlative index. It is worth stressing that fixed base Paasche and Laspeyres indices are very likely to diverge considerably over a five-year period if computers (or any other commodity which has price and quantity trends that are quite different from the trends in the other commodities) are included in the value aggregate under consideration (see Chapter 19 for some "empirical" evidence on this topic).

[75] Again, see Szulc (1983) and Hill (1988).

[76] This formula was implicitly introduced in Törnqvist (1936) and explicitly defined in Törnqvist and Törnqvist (1937).

and Feenstra (1999, p. 61) show that if the logarithmic price ratios $\ln(p_i^t/p_i^{t-1})$ trend linearly with time t and the expenditure shares s_i^t also trend linearly with time, then the Törnqvist index P_T will satisfy the circularity test exactly.[77] Since many economic time series on prices and quantities satisfy these assumptions approximately, the Törnqvist index P_T will satisfy the circularity test approximately. As is seen in Chapter 19, the Törnqvist index generally closely approximates the symmetrically weighted Fisher and Walsh indices, so that for many economic time series (with smooth trends), all three of these symmetrically weighted indices will satisfy the circularity test to a high enough degree of approximation so that it will not matter whether we use the fixed base or chain principle.

15.94 Walsh (1901, p. 401; 1921a, p. 98; 1921b, p. 540) introduced the following useful variant of the circularity test:

$$1 = P(p^0, p^1, q^0, q^1)P(p^1, p^2, q^1, q^2)...P(p^T, p^0, q^T, q^0) \qquad (15.82)$$

The motivation for this test is the following. Use the bilateral index formula $P(p^0,p^1,q^0,q^1)$ to calculate the change in prices going from period 0 to 1, use the same formula evaluated at the data corresponding to periods 1 and 2, $P(p^1,p^2,q^1,q^2)$, to calculate the change in prices going from period 1 to 2, … , use $P(p^{T-1},p^T,q^{T-1},q^T)$ to calculate the change in prices going from period $T-1$ to T, introduce an artificial period T+1 that has exactly the price and quantity of the initial period 0 and use $P(p^T,p^{T+1},q^T,q^{T+1})$ to calculate the change in prices going from period T to T+1. Finally, multiply all of these indices together. Since we end up where we started, the product of all of these indices will ideally be one. Diewert (1993a, p. 40) called this test a *multiperiod identity test*.[78] Note that if $T = 2$ (so that the number of periods is three in total), then Walsh's test reduces to Fisher's (1921, p. 534; 1922, p. 64) time reversal test.[79]

15.95 Walsh (1901, pp. 423-433) showed how his circularity test could be used in order to evaluate how "good" any bilateral index number formula was. What he did was invent

[77] This exactness result can be extended to cover the case when there are monthly proportional variations in prices, and the expenditure shares have constant seasonal effects in addition to linear trends; see Alterman, Diewert and Feenstra (1999, p. 65).

[78] Walsh (1921a, p. 98) called his test the *circular test*,but since Fisher also used this term to describe his transitivity test defined earlier by equation (15.77), it seems best to stick to Fisher's terminology since it is well established in the literature.

[79] Walsh (1921b, pp. 540-541) noted that the time reversal test was a special case of his circularity test.

artificial price and quantity data for five periods, and he added a sixth period that had the data of the first period. He then evaluated the right-hand side of equation (15.82) for various formulae, $P(p^0,p^1,q^0,q^1)$, and determined how far from unity the results were. His "best" formulae had products that were close to one.[80]

15.96 This same framework is often used to evaluate the efficacy of chained *indices* versus their direct counterparts. Thus if the right-hand side of equation (15.82) turns out to be different from unity, the chained indices are said to suffer from "chain drift". If a formula does suffer from chain drift, it is sometimes recommended that fixed base indices be used in place of chained ones. However, this advice, if accepted, would *always* lead to the adoption of fixed base indices, provided that the bilateral index formula satisfies the identity test, $P(p^0,p^0,q^0,q^0) = 1$. Thus it is not recommended that Walsh's circularity test be used to decide whether fixed base or chained indices should be calculated. It is fair to use Walsh's circularity test, as he originally used it as an approximate method for deciding how "good" a particular index number formula is. To decide whether to chain or use fixed base indices, look at how similar the observations being compared are and choose the method which will best link up the most similar observations.

15.97 Various properties, axioms or tests that an index number formula could satisfy have been introduced in this chapter. In the following chapter, the test approach to index number theory is studied in a more systematic manner.

Appendix 15.1 The relationship between the Paasche and Laspeyres indices

1. Recall the notation used in paragraphs 15.11 to 15.17, above. Define the *i*th relative price or price relative r_i and the *i*th quantity relative t_i as follows:

$$r_i \equiv \frac{p_i^1}{p_i^0}; \quad t_i \equiv \frac{q_i^1}{q_i^0}; \qquad i = 1,...,n \qquad \text{(A15.1.1)}$$

Using formula (15.8) for the Laspeyres price index P_L and definitions (A15.1.1), we have:

[80] This is essentially a variant of the methodology that Fisher (1922, p- 284) used to check how well various formulae corresponded to his version of the circularity test.

$$P_L = \sum_{i=1}^{n} r_i s_i^0 \equiv r^* \qquad \text{(A15.1.2)}$$

i.e., we define the "average" price relative r^* as the base period expenditure share-weighted average of the individual price relatives, r_i .

2. Using formula (15.6) for the Paasche price index P_P, we have:

$$P_P \equiv \frac{\sum_{i=1}^{n} p_i^1 q_i^1}{\sum_{m=1}^{n} p_m^0 q_m^1} = \frac{\sum_{i=1}^{n} r_i t_i p_i^0 q_i^0}{\sum_{m=1}^{n} t_m p_m^0 q_m^0} \qquad \text{using definitions (A15.1.1)}$$

$$= \frac{\sum_{i=1}^{n} r_i t_i s_i^0}{\sum_{m=1}^{n} t_m s_m^0} = \left\{ \frac{1}{\sum_{m=1}^{n} t_m s_m^0} \sum_{i=1}^{n} (r_i - r^*)(t_i - t^*) s_i^0 \right\} + r^* \qquad \text{(A15.1.3)}$$

using (A15.1.2) and $\sum_{i=1}^{n} s_i^0 = 1$ and where the "average" quantity relative t^* is defined as

$$t^* \equiv \sum_{i=1}^{n} t_i s_i^0 = Q_L \qquad \text{(A15.1.4)}$$

where the last equality follows using equation (15.11), the definition of the Laspeyres quantity index Q_L.

3. Taking the difference between P_P and P_L and using equations (A15.1.2)–(A15.1.4) yields:

$$P_P - P_L = \frac{1}{Q_L} \sum_{i=1}^{n} (r_i - r^*)(t_i - t^*) s_i^0 \qquad \text{(A15.1.5)}$$

Now let r and t be discrete random variables that take on the n values r_i and t_i respectively. Let s_i^0 be the joint probability that $r = r_i$ and $t = t_i$ for $i = 1,\ldots,n$ and let the joint probability be 0 if $r = r_i$ and $t = t_j$ where $i \neq j$. It can be verified that the summation $\sum_{i=1}^{n} (r_i - r^*)(t_i - t^*) s_i^0$ on the right-hand side of equation (A15.1.5) is the covariance between the price relatives r_i and the corresponding quantity relatives t_i. This covariance can be converted into a correlation coefficient.[81] If this covariance is negative, which is the usual case in the consumer context, then P_P will be less than P_L.

Appendix 15.2 The relationship between the Lowe and Laspeyres indices

1. Recall the notation used in paragraphs 15.33 to 15.48, above. Define the *i*th relative price relating the price of commodity *i* of month *t* to month 0, r_i, and the *i*th quantity relative, t_i, relating quantity of commodity *i* in base year *b* to month 0 t_i as follows:

$$r_i \equiv \frac{p_i^t}{p_i^0} \; t_i \equiv \frac{q_i^b}{q_i^0}; \quad i = 1,\ldots, n \qquad \text{(A15.2.1)}$$

As in Appendix A15.1, the Laspeyres price index $P_L(p^0,p^t,q^0)$ can be defined as r^*, the month 0 expenditure share-weighted average of the individual price relatives r_i defined in (A15.2.1) except that the month *t* price, p_i^t, now replaces period 1 price, p_i^1, in the definition of the *i*th price relative r_i:

$$r^* \equiv \sum_{i=1}^{n} r_i s_i^0 = P_L \qquad \text{(A15.2.2)}$$

2. The “average” quantity relative t^* relating the quantities of base year *b* to those of month 0 is defined as the month 0 expenditure share-weighted average of the individual quantity relatives t_i, defined in (A15.2.1):

$$t^* \equiv \sum_{i=1}^{n} t_i s_i^0 = Q_L \qquad \text{(A15.2.3)}$$

where $Q_L = Q_L(q^0,q^b,p^0)$ is the Laspeyres quantity index relating the quantities of month 0, q^0, to those of the year *b*, q^b, using the prices of month 0, p^0, as weights.

3. Using definition (15.26), the Lowe index comparing the prices in month *t* to those of month 0, using the quantity weights of the base year *b*, is equal to:

[81] See Bortkiewicz (1923, pp. 374-375) for the first application of this correlation coefficient decomposition technique.

$$P_{Lo}(p^0, p^t, q^b) \equiv \frac{\sum_{i=1}^{n} p_i^t q_i^b}{\sum_{i=1}^{n} p_i^0 q_i^b} = \frac{\sum_{i=1}^{n} p_i^t t_i q_i^0}{\sum_{i=1}^{n} p_i^0 t_i q_i^0} \qquad \text{using (A15.2.1)}$$

$$= \left\{\frac{\sum_{i=1}^{n} p_i^t t_i q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0}\right\}\left\{\frac{\sum_{i=1}^{n} p_i^0 t_i q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0}\right\}^{-1}$$

$$= \left\{\frac{\sum_{i=1}^{n} \left(\frac{p_i^t}{p_i^0}\right) t_i p_i^0 q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0}\right\} \Big/ t^* \quad \text{using (A15.2.3)}$$

$$= \left\{\frac{\sum_{i=1}^{n} r_i t_i p_i^0 q_i^0}{\sum_{i=1}^{n} p_i^0 q_i^0}\right\} \Big/ t^* \quad \text{using (A15.2.1)}$$

$$= \frac{\sum_{i=1}^{n} r_i t_i s_i^0}{t^*} = \frac{\sum_{i=1}^{n} (r_i - r^*) t_i s_i^0}{t^*} + \frac{\sum_{i=1}^{n} r^* t_i s_i^0}{t^*}$$

$$= \frac{\sum_{i=1}^{n} (r_i - r^*) t_i s_i^0}{t^*} + \frac{r^*\left[\sum_{i=1}^{n} t_i s_i^0\right]}{t^*}$$

$$= \frac{\sum_{i=1}^{n} (r_i - r^*) t_i s_i^0}{t^*} + \frac{r^*\left[t^*\right]}{t^*} \qquad \text{using (A15.2.3)}$$

$$= \frac{\sum_{i=1}^{n} (r_i - r^*)(t_i - t^*) s_i^0}{t^*} + \frac{\sum_{i=1}^{n} (r_i - r^*) t^* s_i^0}{t^*} + r^*$$

$$= \frac{\sum_{i=1}^{n} (r_i - r^*)(t_i - t^*) s_i^0}{t^*} + \frac{t^*\left[\sum_{i=1}^{n} r_i s_i^0 - r^*\right]}{t^*} + r^*$$

$$= \frac{\sum_{i=1}^{n} (r_i - r^*)(t_i - t^*) s_i^0}{t^*} + r^* \quad \text{since} \sum_{\mathrm{i}=1}^{\mathrm{n}} r_i s_i^0 = r^*$$

$$= P_L(p^0, p^t, q^0) + \frac{\sum_{i=1}^{n} (r_i - r^*)(t_i - t^*) s_i^0}{Q_L(q^0, q^b, p^0)} \qquad \text{(A15.2.4)}$$

since using (A15.2.2), r^* equals the Laspeyres price index, $P_L(p^0,p^t,q^0)$, and using (A15.2.3), t^* equals the Laspeyres quantity index, $Q_L(q^0,q^b,p^0)$. Thus equation (A15.2.4) tells us that the Lowe price index using the quantities of year b as weights, $P_{Lo}(p^0,p^t,q^b)$, is equal to the usual Laspeyres index using the quantities of month 0 as weights, $P_L(p^0,p^t,q^0)$, plus a covariance term $\sum_{i=1}^{n}(r_i - r^*)(t_i - t^*)s_i^0$ between the price relatives $r_i \equiv p_i^t/p_i^0$ and the quantity relatives $t_i \equiv q_i^b/q_i^0$, divided by the Laspeyres quantity index $Q_L(q^0,q^b,p^0)$ between month 0 and base year b.

Appendix 15.3 The relationship between the Young index and its time antithesis

1. Recall that the direct Young index, $P_Y(p^0,p^t,s^b)$, was defined by equation (15.48) and its time antithesis, $P_Y^*(p^0,p^t,s^b)$, was defined by equation (15.52). Define the ith relative price between months 0 and t as

$$r_i \equiv p_i^t / p_i^0; \qquad i = 1,\ldots,n \qquad \text{(A15.3.1)}$$

and define the weighted average (using the base year weights s_i^b) of the r_i as

$$r^* \equiv \sum_{i=}^{n} s_i^b r_i \qquad \text{(A15.3.2)}$$

which turns out to equal the direct Young index, $P_Y(p^0,p^t,s^b)$. Define the deviation e_i of r_i from their weighted average r^* using the following equations:

$$r_i = r^*(1+e_i); \qquad i = 1,\ldots,n \qquad \text{(A15.3.3)}$$

If equation (A15.3.3) is substituted into equation (A15.3.2), the following equation is obtained:

$$\begin{aligned} r^* &\equiv \sum_{i=1}^{n} s_i^b r^*(1+e_i) \\ &= r^* + r^*\sum_{i=1}^{n} s_i^b e_i \qquad \text{since } \sum_{i=1}^{n} s_i^b = 1 \end{aligned} \qquad \text{(A15.3.4)}$$

$$e^* \equiv \sum_{i=1}^{n} s_i^b e_i = 0 \qquad \text{(A15.3.5)}$$

Thus the weighted mean e^* of the deviations e_i equals 0.

2. The direct Young index, $P_Y(p^0,p^t,s^b)$, and its time antithesis, $P_Y{}^*(p^0,p^t,s^b)$, can be written as functions of r^*, the weights s_i^b and the deviations of the price relatives e_i as follows:

$$P_Y(p^0,p^t,s^b) = r^* \qquad \text{(A15.3.6)}$$

$$\begin{aligned} P_Y^*(p^0,p^t,s^b) &= \left[\sum_{i=1}^{n} s_i^b \left\{r^*(1+e_i)\right\}^{-1}\right]^{-1} \\ &= r^*\left[\sum_{i=1}^{n} s_i^b (1+e_i)^{-1}\right]^{-1} \end{aligned} \qquad \text{(A15.3.7)}$$

3. Now regard $P_Y{}^*(p^0,p^t,s^b)$ as a function of the vector of deviations, $e \equiv [e_1,\ldots,e_n]$, say $P_Y{}^*(e)$. The second-order Taylor series approximation to $P_Y{}^*(e)$ around the point $e = 0_n$ is given by the following expression:[82]

$$\begin{aligned} P_Y^*(e) &\approx r^* + r^*\sum_{i=1}^{n} s_i^b e_i + r^*\sum_{i=1}^{n}\sum_{j=1}^{n} s_i^b s_j^b e_i e_j - r^*\sum_{i=1}^{n} s_i^b [e_i]^2 \\ &= r^* + r^* 0 + r^*\sum_{i=1}^{n} s_i^b\left[\sum_{j=1}^{n} s_j^b e_j\right] e_i - r^*\sum_{i=1}^{n} s_i^b\left[e_i - e^*\right]^2 \quad \text{using (A15.3.5)} \\ &= r^* + r^*\sum_{i=1}^{n} s_i^b [0] e_i - r^*\sum_{i=1}^{n} s_i^b\left[e_i - e^*\right]^2 \quad \text{using (A15.3.5)} \\ &= P_Y(p^0,p^t,s^b) - P_Y(p^0,p^t,s^b)\sum_{i=1}^{n} s_i^b\left[e_i - e^*\right]^2 \quad \text{using (A15.3.6)} \\ &= P_Y(p^0,p^t,s^b) - P_Y(p^0,p^t,s^b)\,\mathrm{Var}\, e \end{aligned} \qquad \text{(A15.3.8)}$$

where the weighted sample variance of the vector e of price deviations is defined as

$$\mathrm{Var}\; e \;\equiv \sum_{i=1}^{n} s_i^b\left[e_i - e^*\right]^2 \qquad \text{(A15.3.9)}$$

4. Rearranging equation (A15.3.8) gives the following approximate relationship between the direct Young index $P_Y(p^0,p^t,s^b)$ and its time antithesis $P_Y{}^*(p^0,p^t,s^b)$, to the accuracy of a second-

[82] This type of second order approximation is attributable to Dalén (1992; 143) for the case $r^* =1$ and to Diewert (1995a, p. 29) for the case of a general r^*.

order Taylor series approximation about a price point where the month t price vector is proportional to the month 0 price vector:

$$P_Y(p^0, p^t, s^b) \approx P_Y^*(p^0, p^t, s^b) + P_Y(p^0, p^t, s^b) \text{ Var } e \qquad \text{(A15.3.10)}$$

Thus, to the accuracy of a second-order approximation, the direct Young index will exceed its time antithesis by a term equal to the direct Young index times the weighted variance of the deviations of the price relatives from their weighted mean. Thus the bigger is the dispersion in relative prices, the more the direct Young index will exceed its time antithesis.

Appendix 15.4 The relationship between the Divisia and economic approaches

1. Divisia's approach to index number theory relied on the theory of differentiation. Thus it does not appear to have any connection with economic theory. However, starting with Ville (1946), a number of economists[83] have established that the Divisia price and quantity indices *do* have a connection with the economic approach to index number theory. This connection is outlined in this appendix.

2. The economic approach to the determination of the price level and the quantity level is first outlined. The particular economic approach that is used here is attributable to Shephard (1953; 1970), Samuelson (1953) and Samuelson and Swamy (1974).

3. It is assumed that "the" consumer has well-defined *preferences* over different combinations of the n consumer commodities or items. Each combination of items can be represented by a positive vector $q \equiv [q_1,\ldots,q_n]$. The consumer's preferences over alternative possible consumption vectors q are assumed to be representable by a continuous, non-decreasing and concave utility function f. It is further assumed that the consumer minimizes the cost of achieving the period t utility level $u^t \equiv f(q^t)$ for periods $t = 0,1,\ldots,T$. Thus it is assumed that the observed period t consumption vector q^t solves the following period t cost minimization problem:

[83] See for example Malmquist (1953, p. 227), Wold (1953, pp. 134-147), Solow (1957), Jorgenson and Griliches (1967) and Hulten (1973), and see Balk (2000a) for a recent survey of work on Divisia price and quantity indices.

$$C(u^t, p^t) \equiv \min_q \left\{ \sum_{i=1}^{n} p_i^t q_i : f(q) = u^t = f(q^t) \right\}$$
$$= \sum_{i=1}^{n} p_i^t q_i^t; \quad t = 0,1,...,T \qquad \text{(A15.4.1)}$$

The period t price vector for the n commodities under consideration that the consumer faces is p^t. Note that the solution to the period t cost or expenditure minimization problem defines the *consumer's cost function*, $C(u^t,p^t)$.

4. An additional regularity condition is placed on the consumer's utility function f. It is assumed that f is (positively) linearly homogeneous for strictly positive quantity vectors. Under this assumption, the consumer's expenditure or cost function, $C(u,p)$, decomposes into $uc(p)$ where $c(p)$ is the consumer's unit cost function.[84] The following equation is obtained:

$$\sum_{i=1}^{n} p_i^t q_i^t = c(p^t) f(q^t) \quad \text{for } t = 0,1,\ldots,T \qquad \text{(A15.4.2)}$$

Thus the period t total expenditure on the n commodities in the aggregate, $\sum_{i=1}^{n} p_i^t q_i^t$,

decomposes into the product of two terms, $c(p^t)f(q^t)$. The period t unit cost, $c(p^t)$, can be identified as the period t price level P^t and the period t level of utility, $f(q^t)$, can be identified as the period t quantity level Q^t.

5. The economic price level for period t, $P^t \equiv c(p^t)$, defined in the previous paragraph, is now related to the Divisia price level for time t, $P(t)$, that was implicitly defined by the differential equation (15.67). As in paragraphs 15.65 to 15.71, think of the prices as being continuous, differentiable functions of time, $p_i(t)$ say, for $i = 1,\ldots, n$. Thus the unit cost function can be regarded as a function of time t as well; i.e., define the unit cost function as a function of t as

$$c^*(t) \equiv c\left[p_1(t), p_2(t),..., p_n(t)\right] \qquad \text{(A15.4.3)}$$

6. Assuming that the first-order partial derivatives of the unit cost function $c(p)$ exist,

[84] See Diewert (1993b, pp.120-121) for material on unit cost functions. This material will also be covered in Chapter 17.

calculate the logarithmic derivative of $c^*(t)$ as follows:

$$\frac{d\ln c^*(t)}{dt} \equiv \frac{1}{c^*(t)}\frac{dc^*(t)}{dt}$$
$$= \frac{\sum_{i=1}^{n} c_i\left[p_1(t), p_2(t),..., p_n(t)\right] p_i^{'}(t)}{c\left[p_1(t), p_2(t),..., p_n(t)\right]} \qquad \text{(A15.4.4)}$$

where $c_i[p_1(t),p_2(t),\ldots,p_n(t)] \equiv \partial c[p_1(t),p_2(t),\ldots,p_n(t)]/\partial p_i$ is the partial derivative of the unit cost function with respect to the ith price, p_i, and $p_i'(t) \equiv dp_i(t)/dt$ is the time derivative of the ith price function, $p_i(t)$. Using Shephard's (1953, p. 11) Lemma, the consumer's cost-minimizing demand for commodity i at time t is:

$$q_i(t) = u(t)c_i\left[p_1(t), p_2(t),..., p_n(t)\right] \text{ for } i = 1,..., n \qquad \text{(A15.4.5)}$$

where the utility level at time t is $u(t) = f[q_1(t),q_2(t),\ldots,q_n(t)]$. The continuous time counterpart to equations (A15.4.2) above is that total expenditure at time t is equal to total cost at time t which in turn is equal to the utility level, $u(t)$, times the period t unit cost, $c^*(t)$:

$$\sum_{i=1}^{n} p_i(t)q_i(t) = u(t)c^*(t) = u(t)c\left[p_1(t), p_2(t),..., p_n(t)\right]. \qquad \text{(A15.4.6)}$$

7. The logarithmic derivative of the Divisia price level $P(t)$ can be written as (recall equation (15.67) above):

$$\frac{P'(t)}{P(t)} = \frac{\sum_{i=1}^{n} p_i'(t) q_i(t)}{\sum_{i=1}^{n} p_i(t) q_i(t)} = \frac{\sum_{i=1}^{n} p_i'(t) q_i(t)}{u(t)c^*(t)} \text{ using (A15.4.6)}$$

$$= \frac{\sum_{i=1}^{n} p_i'(t)\{u(t)c[p_1(t), p_2(t),..., p_n(t)]\}}{u(t)c^*(t)} \text{ using (A15.4.5)}$$

$$= \frac{\sum_{i=1}^{n} c_i[p_1(t), p_2(t),..., p_n(t)]p_i'(t)}{c^*(t)} = \frac{1}{c^*(t)} \frac{dc^*(t)}{dt} \text{ using (A15.4.4)}$$

$$\equiv \frac{c^{*}{}'(t)}{c^*(t)}. \qquad \text{(A15.4.7)}$$

Thus under the above continuous time cost-minimizing assumptions, the Divisia price level, $P(t)$, is essentially equal to the unit cost function evaluated at the time t prices, $c^*(t) \equiv c[p_1(t),p_2(t),\ldots,p_n(t)]$.

8. If the Divisia price level $P(t)$ is set equal to the unit cost function $c^*(t) \equiv c[p_1(t),p_2(t),\ldots,p_n(t)]$, then from equation (A15.4.2), it follows that the Divisia quantity level $Q(t)$ defined by equation (15.68) will equal the consumer's utility function regarded as a function of time, $f^*(t) \equiv f[q_1(t),\ldots,q_n(t)]$. Thus, under the assumption that the consumer is continuously minimizing the cost of achieving a given utility level where the utility or preference function is linearly homogeneous, it has been shown that the Divisia price and quantity levels $P(t)$ and $Q(t)$, defined implicitly by the differential equations (15.67) and (15.68), are essentially equal to the consumer's unit cost function $c^*(t)$ and utility function $f^*(t)$ respectively.[85] These are rather remarkable equalities since in principle, given the functions of time, $p_i(t)$ and $q_i(t)$, the differential equations that define the Divisia price and quantity indices can be solved numerically and hence $P(t)$ and $Q(t)$ are in principle observable (up to some normalizing constants).

[85] Obviously, the scale of the utility and cost functions are not uniquely determined by the differential equations (15.62) and (15.63).

9. For more on the Divisia approach to index number theory, see Vogt (1977; 1978) and Balk (2000a). An alternative approach to Divisia indices using line integrals may be found in the forthcoming companion volume *Producer price index manual* (IMF et al., 2004).